Lecture 13:

Dynamics and Time Integration

Interactive Computer Graphics
Stanford CS248, Winter 2020

Challenge: hand animate this clothing!

Dynamical description of motion

"A change in motion is proportional to the motive force impressed and takes place along the straight line in which that force is impressed."

—Sir Isaac Newton, 1687

"Dynamics is concerned with the study of forces and their effect on motion, as opposed to kinematics, which studies the motion of objects without reference to its causes."

—Sir Wiki Pedia, present

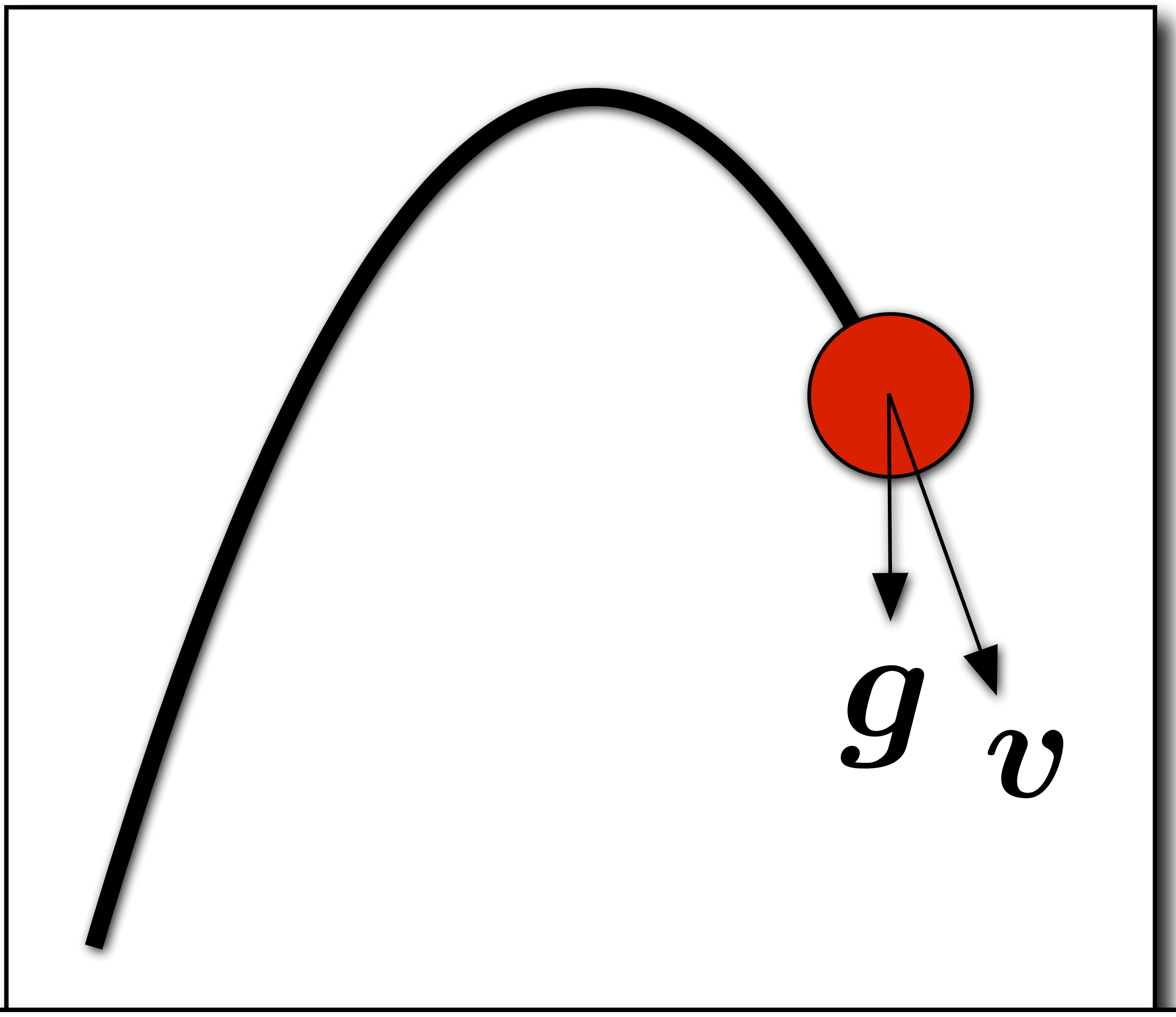
(Q: Is keyframe interpolation *dynamic*, or *kinematic?*)

Newton's 2nd law

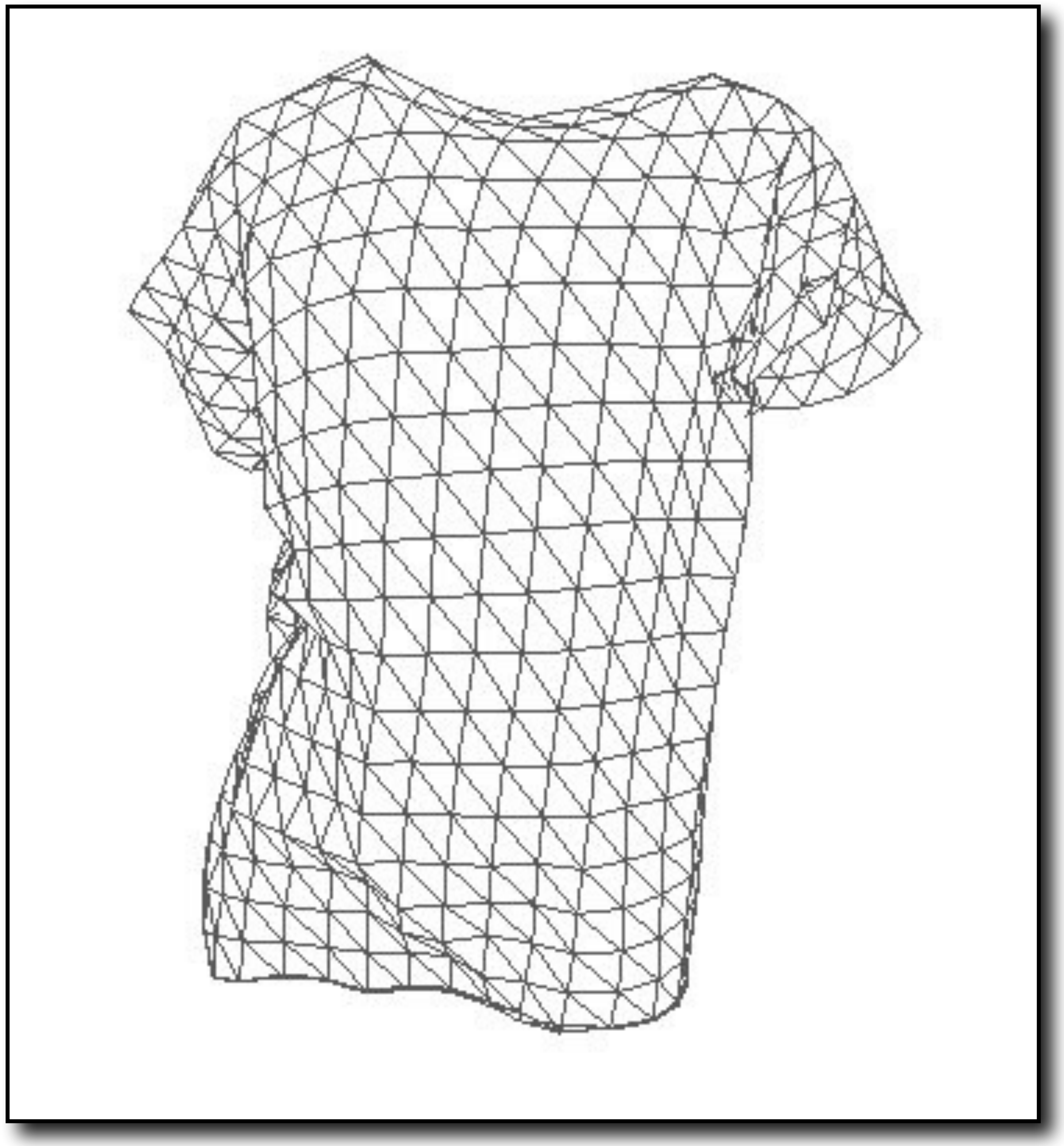
$$\underset{\text{force}}{F} = \overset{\text{mass}}{m}\overset{\text{acceleration}}{a}$$

Physically based animation

- **Generate motion of objects using numerical simulation**



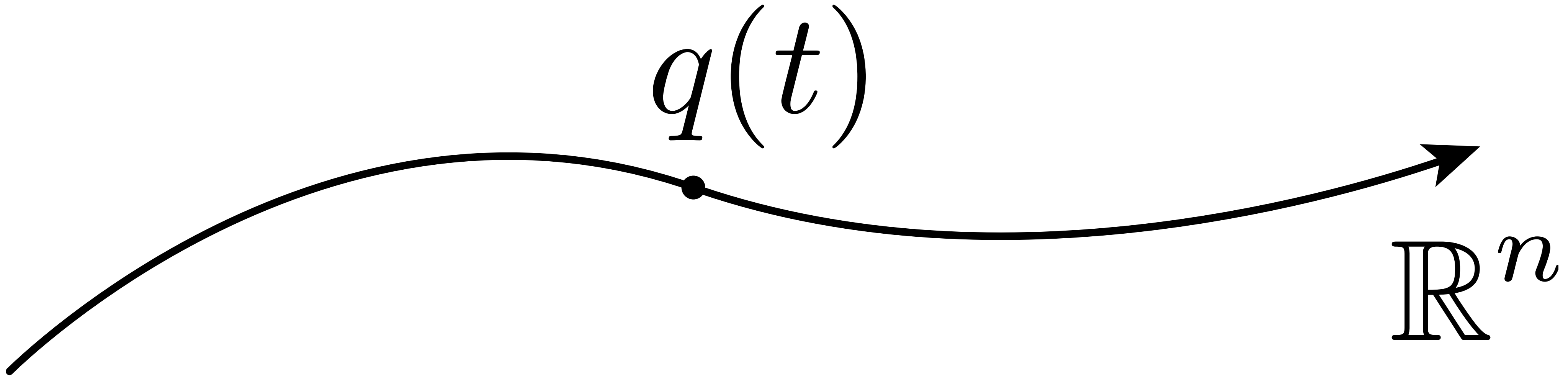
$$\boldsymbol{x}^{t+\Delta t} = \boldsymbol{x}^t + \Delta t \boldsymbol{v}^t + \frac{1}{2}(\Delta t)^2 \boldsymbol{a}^t$$

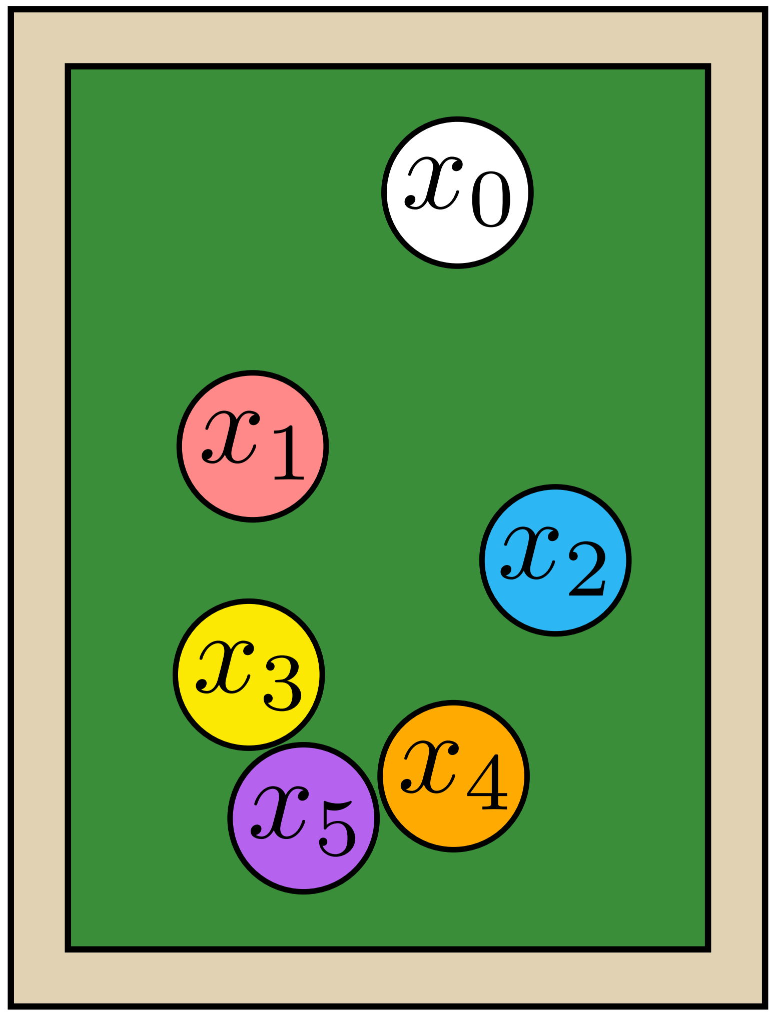


Generalized coordinates

- Often describing systems with many, many moving pieces
- E.g., a collection of billiard balls, each with position x_i
- Collect them all into a single vector of *generalized coordinates:*

$$q = (x_0, x_1, \ldots, x_n)$$



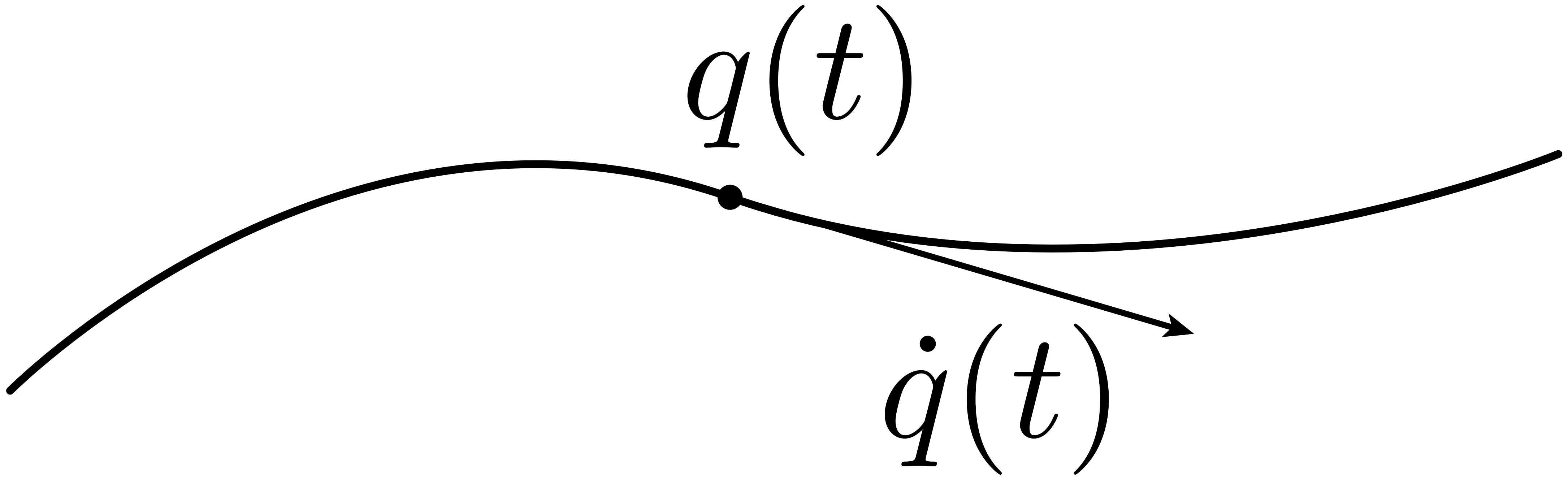


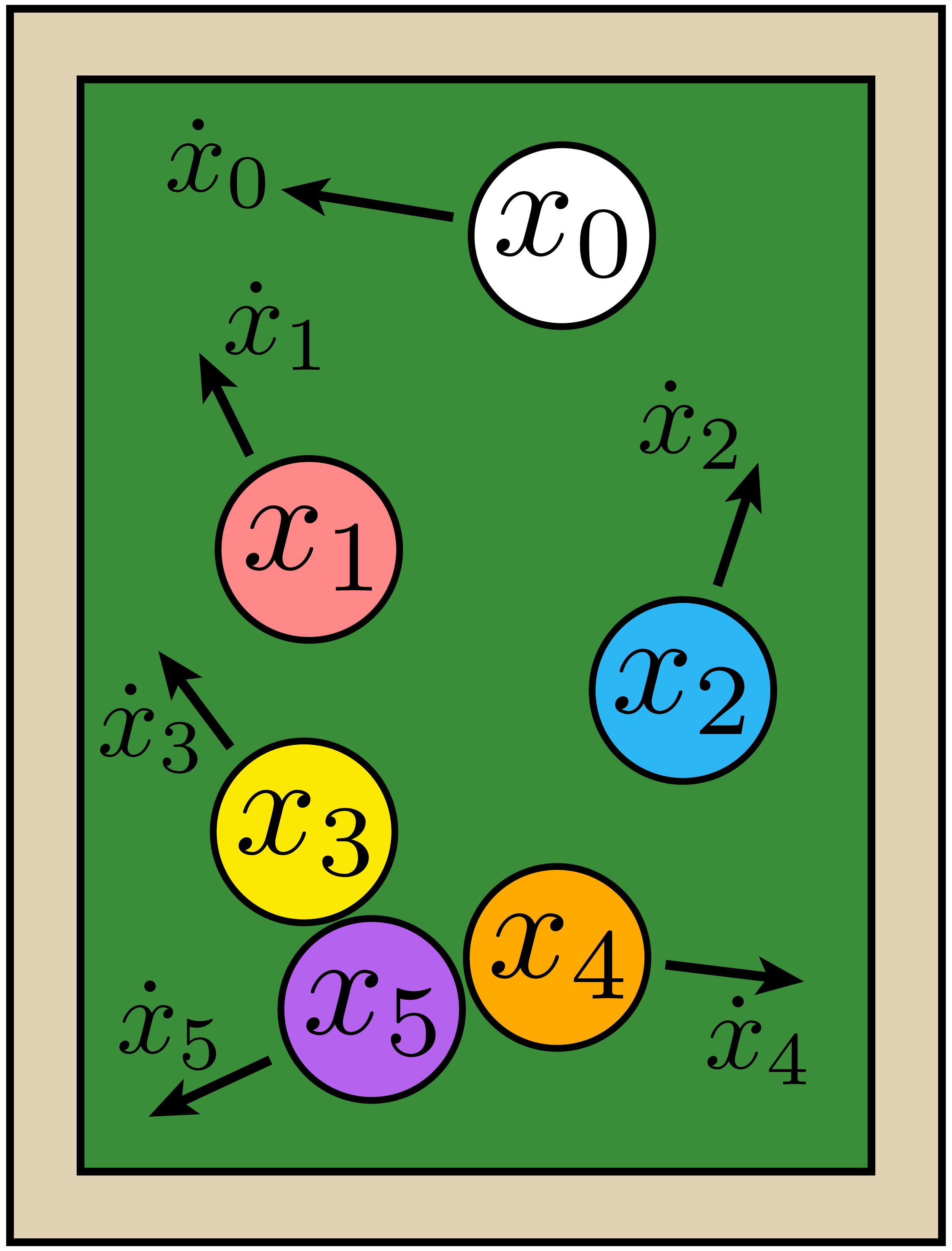
- Can think of q as a *single point* moving along a trajectory in R^n
- This way of thinking naturally maps to the way we actually solve equations on a computer: all variables are often "stacked" into a big long vector and handed to a solver

Generalized velocity

- **Generalized velocity: it's the time derivative of the generalized coordinates!**

$$\dot{q} = (\dot{x}_0, \dot{x}_1, \ldots, \dot{x}_n)$$





All of life (and physics) is just traveling along a curve...

Ordinary differential equations

- Many dynamical systems can be described via an *ordinary differential equation (ODE)* in generalized coordinates:

change in configuration over time

velocity function

$$\frac{d}{dt}q = f(q, \dot{q}, t)$$

- ODE doesn't have to describe mechanical phenomenon, e.g.,

$$\frac{d}{dt}u(t) = au$$

"rate of growth is proportional to value"

- Solution: $u(t) = be^{at}$
- Describes exponential decay ($a < 1$), or really great stock ($a > 1$)
- "Ordinary" means "involves derivatives in time but not space"
- We'll leave talking about spatial derivatives (PDEs) to CS348C

Dynamics via ODEs

- **Another key example: Newton's 2nd law!**

$$\ddot{q} = F/m$$

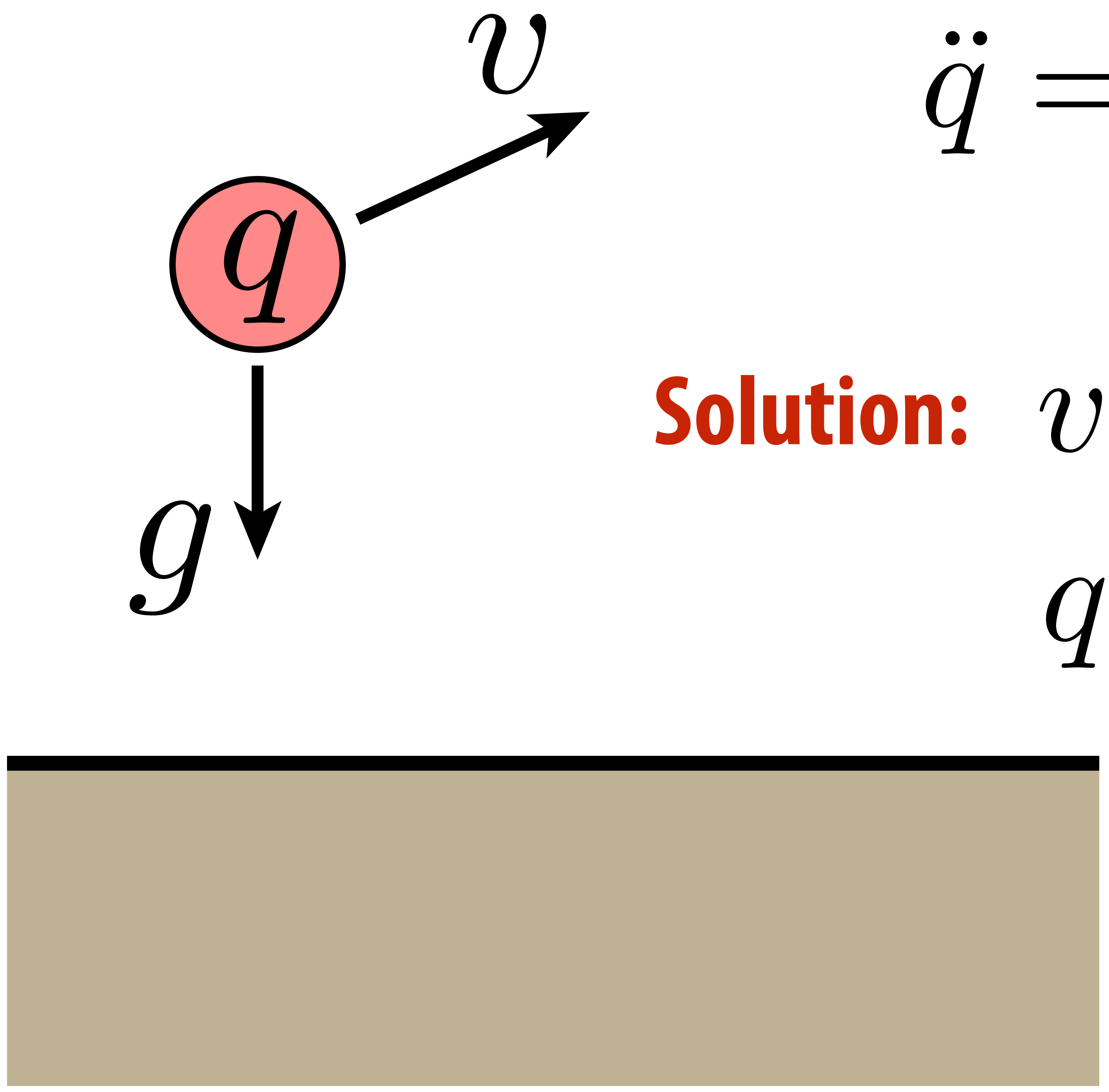
- **"Second order" ODE since we take *two* time derivatives**
- **Can also write as a *system* of two *first order* ODEs, by introducing new "dummy" variable for velocity:**

$$\dot{q} = v$$
$$\dot{v} = F/m$$

- **Splitting things up this way will make it easier to talk about solving these equations numerically**

Simple example: throwing a rock

- **Consider a rock* of mass *m* tossed under force of gravity *g***
- **Easy to write dynamical equations, since only force is gravity:**



$$\ddot{q} = g/m$$

or

$$\begin{aligned} \dot{q} &= v \\ \dot{v} &= g/m \end{aligned}$$

Solution:

$$\begin{aligned} v(t) &= v_0 + \frac{t}{m} g \\ q(t) &= q_0 + t v_0 + \frac{t^2}{2m} g \end{aligned}$$

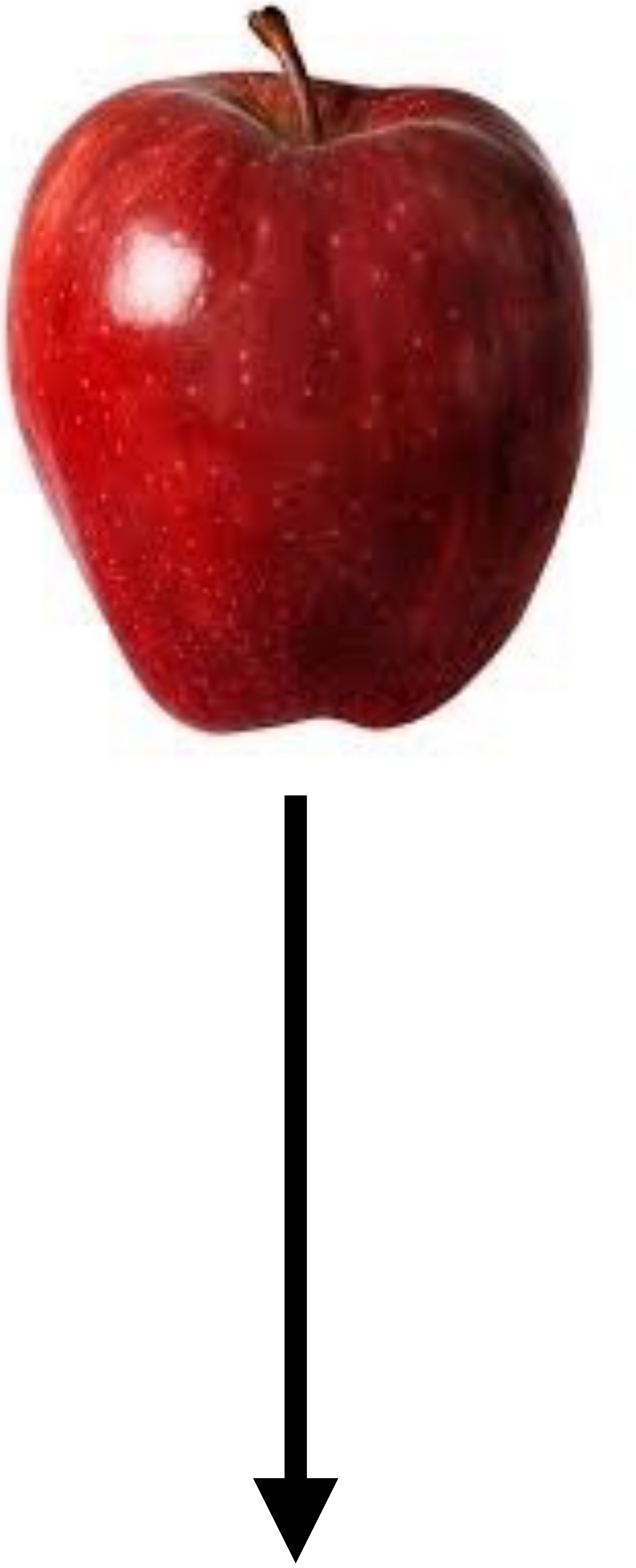
(What do we need a computer for?!)

*Yes, this rock is spherical and has uniform density.

Force due to gravity

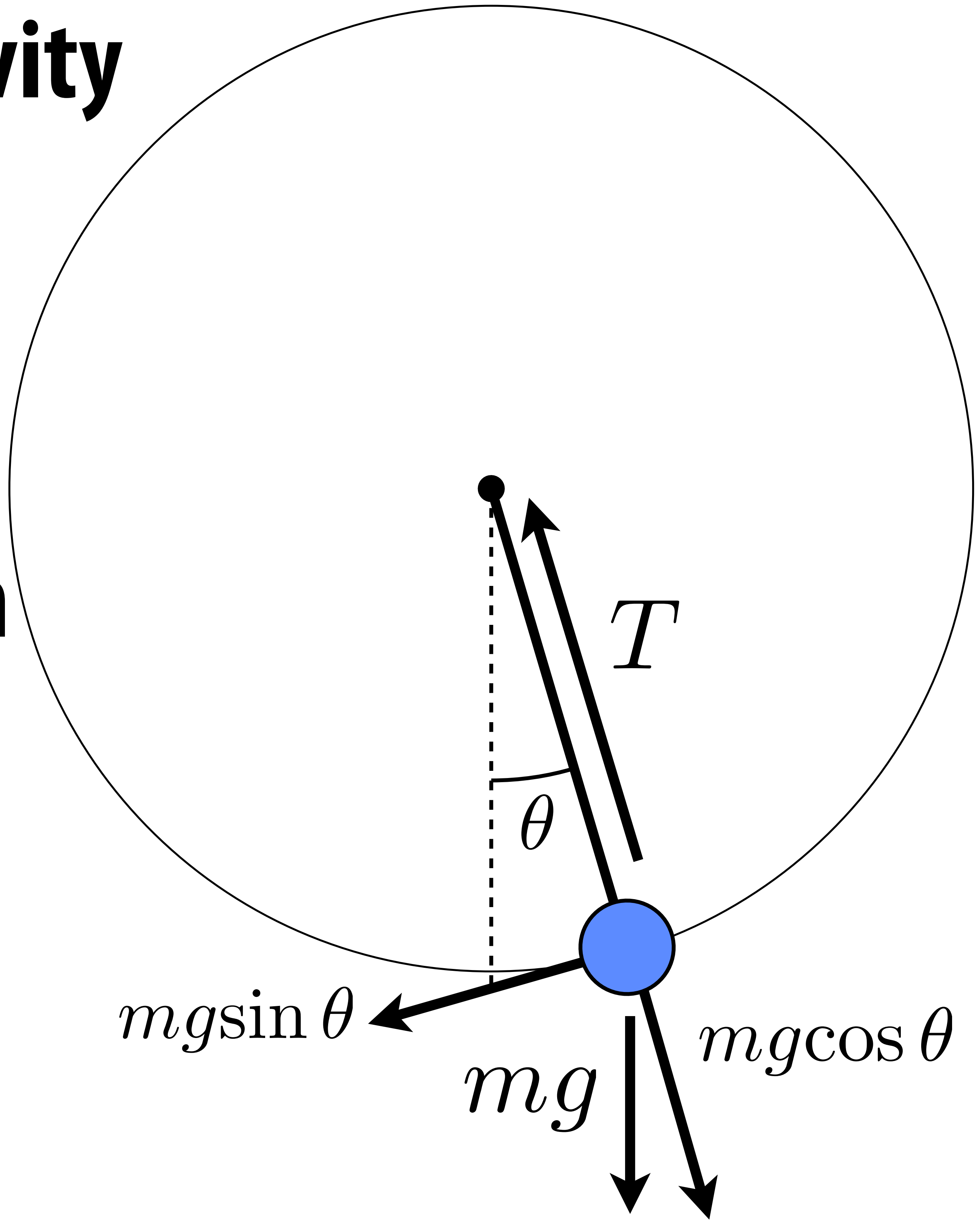
- **Gravity at earth's surface due to earth**
  - ***g*** **is gravitational acceleration,** $g = -9.8 \text{m/s}^2$

$$g = (0, 0, -9.8)\,\mathrm{m/s^2}$$

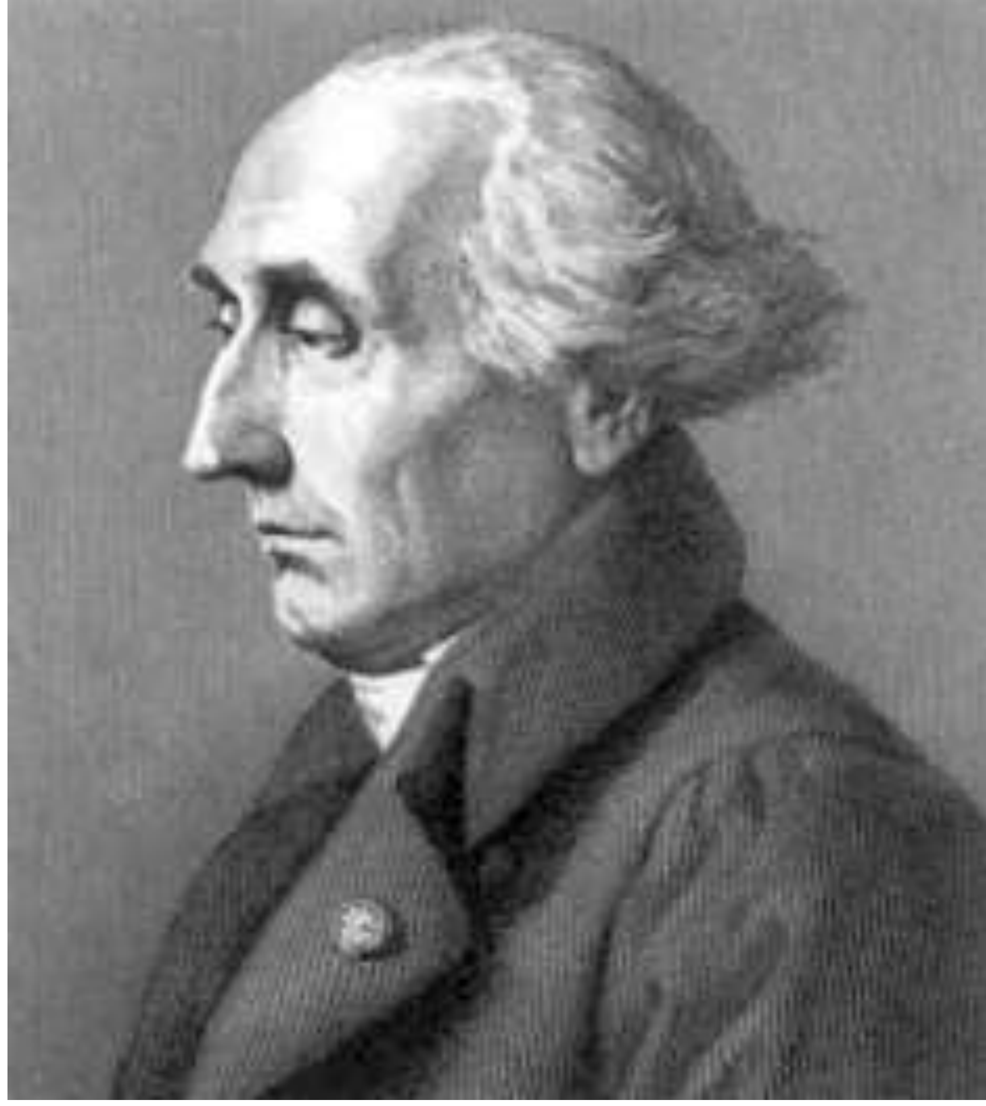


Slightly harder example: pendulum

- Mass on end of a bar, swinging under gravity
- What are the equations of motion?
- Same as "rock" problem, but *constrained*
- Could use a *"force diagram"*
  - You probably did this for many hours in high school/college
  - Let's do something different…



Lagrangian mechanics



Joe Lagrange

- **Beautifully simple recipe:**
  1. Write down kinetic energy K
  2. Write down potential energy U
  3. Write down *Lagrangian* $\mathcal{L} := K - U$
  4. Dynamics of system given by *Euler-Lagrange equation*

becomes (generalized) "MASS TIMES ACCELERATION" →

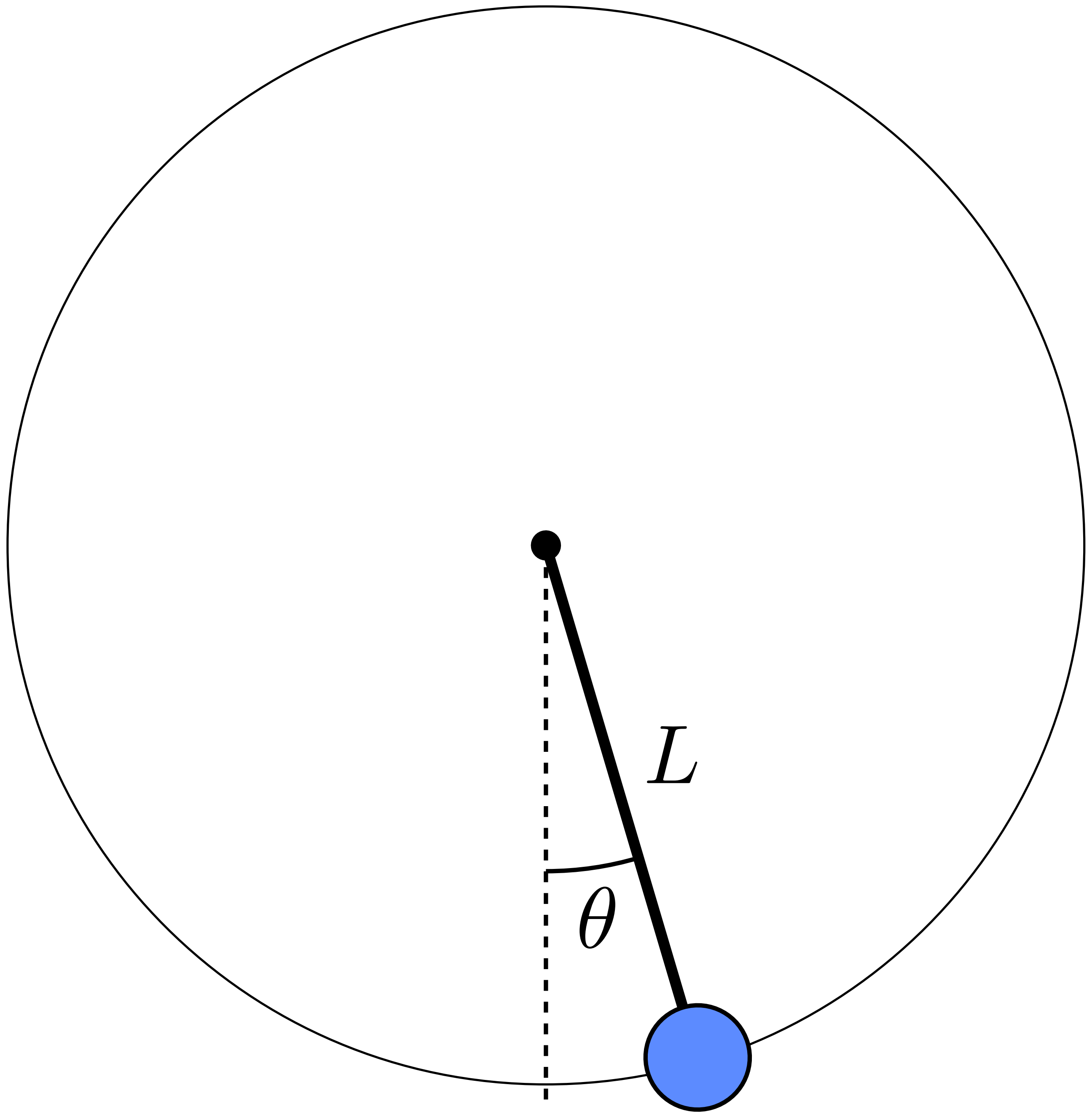
$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial \mathcal{L}}{\partial q}$$

← becomes (generalized) "FORCE"

- **Why is this useful?**
  - often easier to come up with (scalar) energies than forces
  - very general, works in any kind of generalized coordinates
  - helps develop nice class of numerical integrators (symplectic)

Great reference: Sussman & Wisdom, "Structure and Interpretation of Classical Mechanics"

Lagrangian mechanics - example



- **Generalized coordinates for pendulum?**

$$q = \theta$$

just one coordinate:
angle with the vertical direction

- **Kinetic energy (mass *m*)?**

$$K = \tfrac{1}{2}I\omega^2 = \tfrac{1}{2}mL^2\dot{\theta}^2$$

- **Potential energy?**

$$U = mgh = -mgL\cos\theta$$

- **Euler-Lagrange equations:** (from here, just "plug and chug"—even a computer could do it!)

$$\mathcal{L} = K - U = m(\tfrac{1}{2}L^2\dot{\theta}^2 + gL\cos\theta)$$

$$\frac{\partial\mathcal{L}}{\partial\dot{q}} = \frac{\partial\mathcal{L}}{\partial\dot{\theta}} = mL^2\dot{\theta} \qquad \frac{\partial\mathcal{L}}{\partial q} = \frac{\partial\mathcal{L}}{\partial\theta} = -mgL\sin\theta$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}} = \frac{\partial\mathcal{L}}{\partial q} \quad \Rightarrow \quad \boxed{\ddot{\theta} = -\tfrac{g}{L}\sin\theta}$$

Solving the pendulum

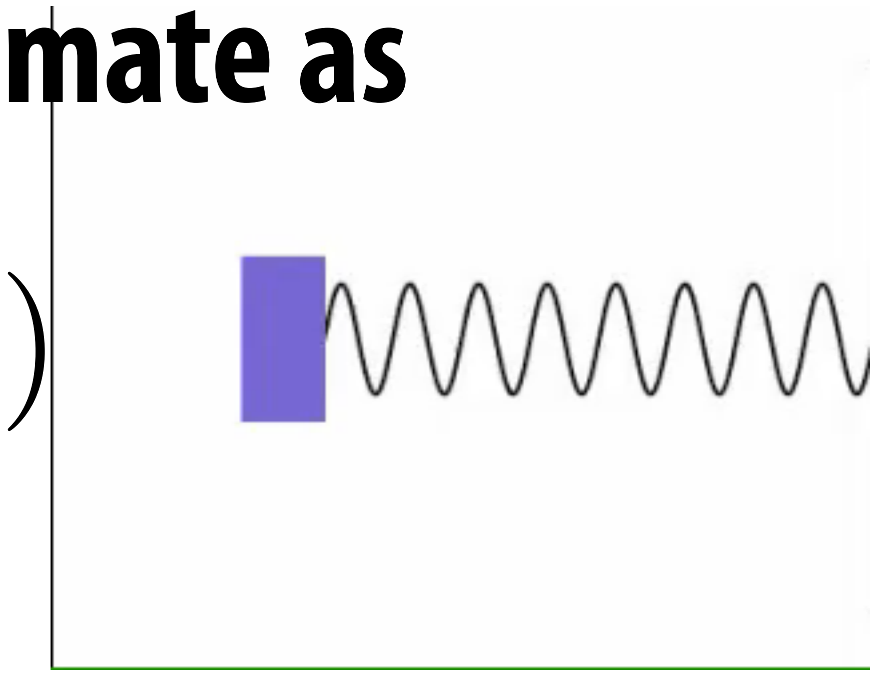
- Great, now we have a nice simple equation for the pendulum:

$$\ddot{\theta} = -\tfrac{g}{L}\sin\theta$$

- For small angles (e.g., clock pendulum) can approximate as

$$\ddot{\theta} = -\tfrac{g}{L}\theta \qquad \Rightarrow \theta(t) = a\cos(t\sqrt{g/L} + b)$$

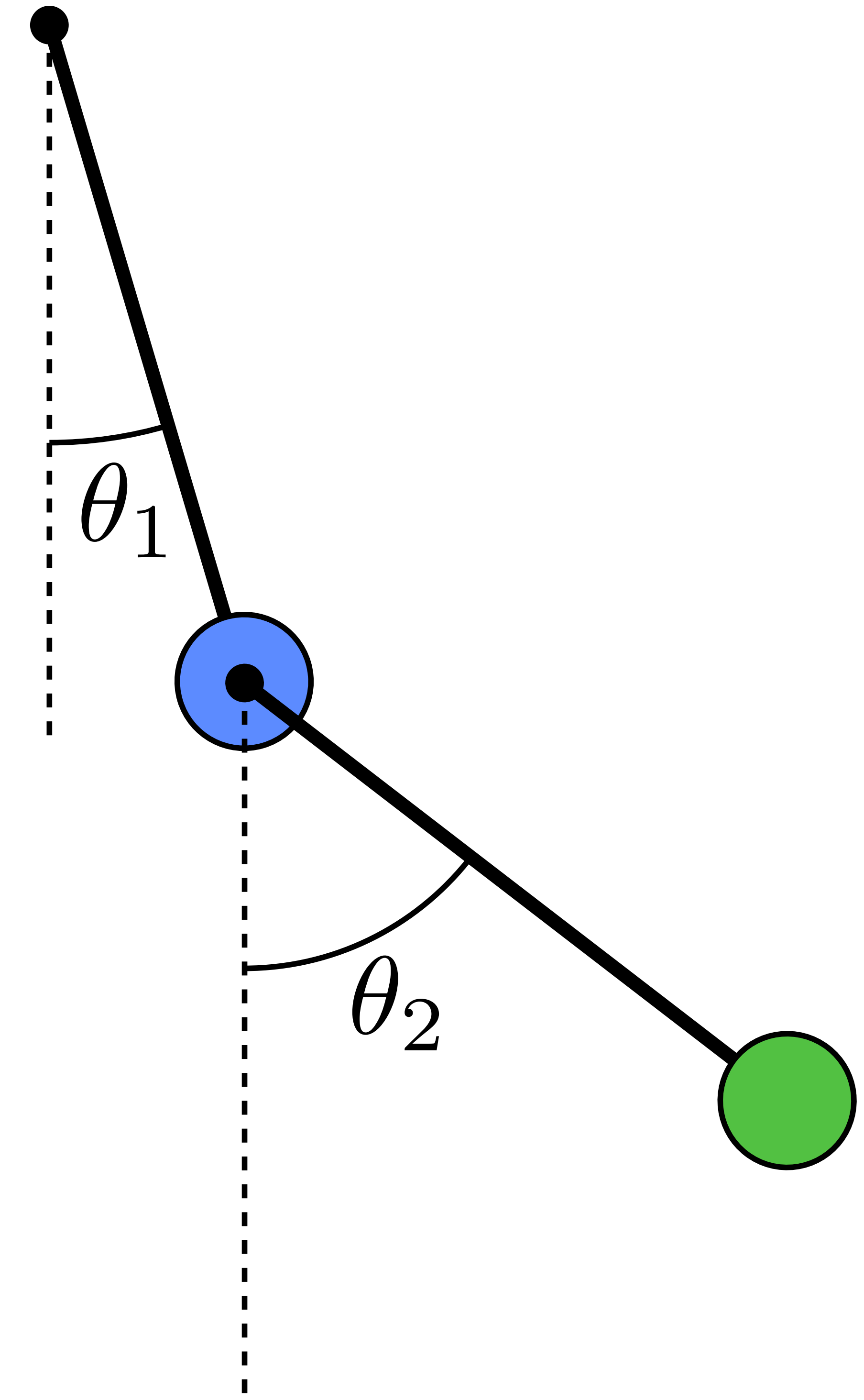
“harmonic oscillator”



- In general, there is *no closed form solution!*
- Hence, we *must* use a numerical approximation
- ...And this was (almost) the simplest system we can think of!
- (What if we want to animate something more interesting?)

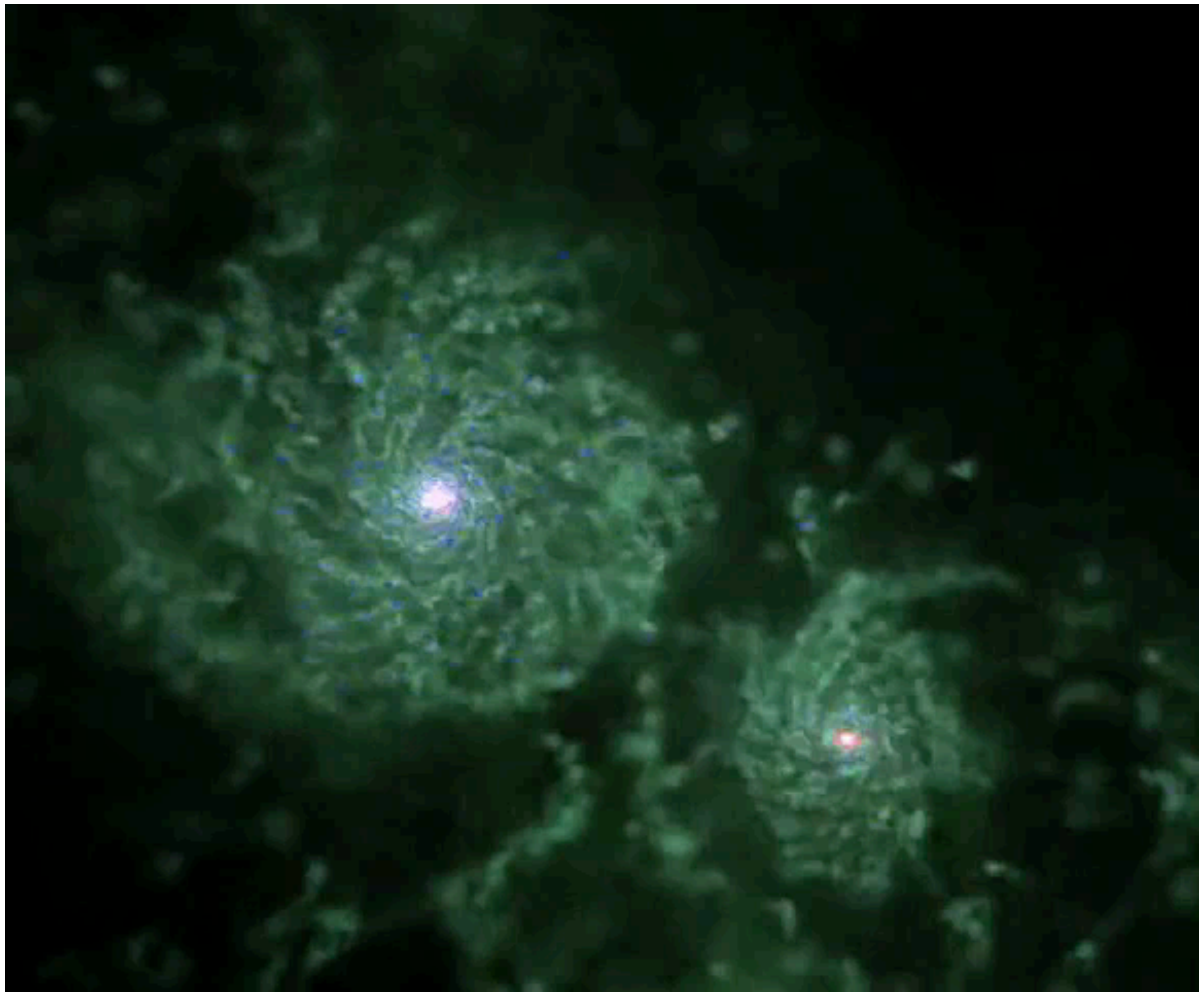
Not-so-simple example: double pendulum

- Blue ball swings from fixed point; green ball swings from blue one
- Simple system… not-so-simple motion!
- Chaotic: perturb input, wild changes to output
- Must again use numerical approximation

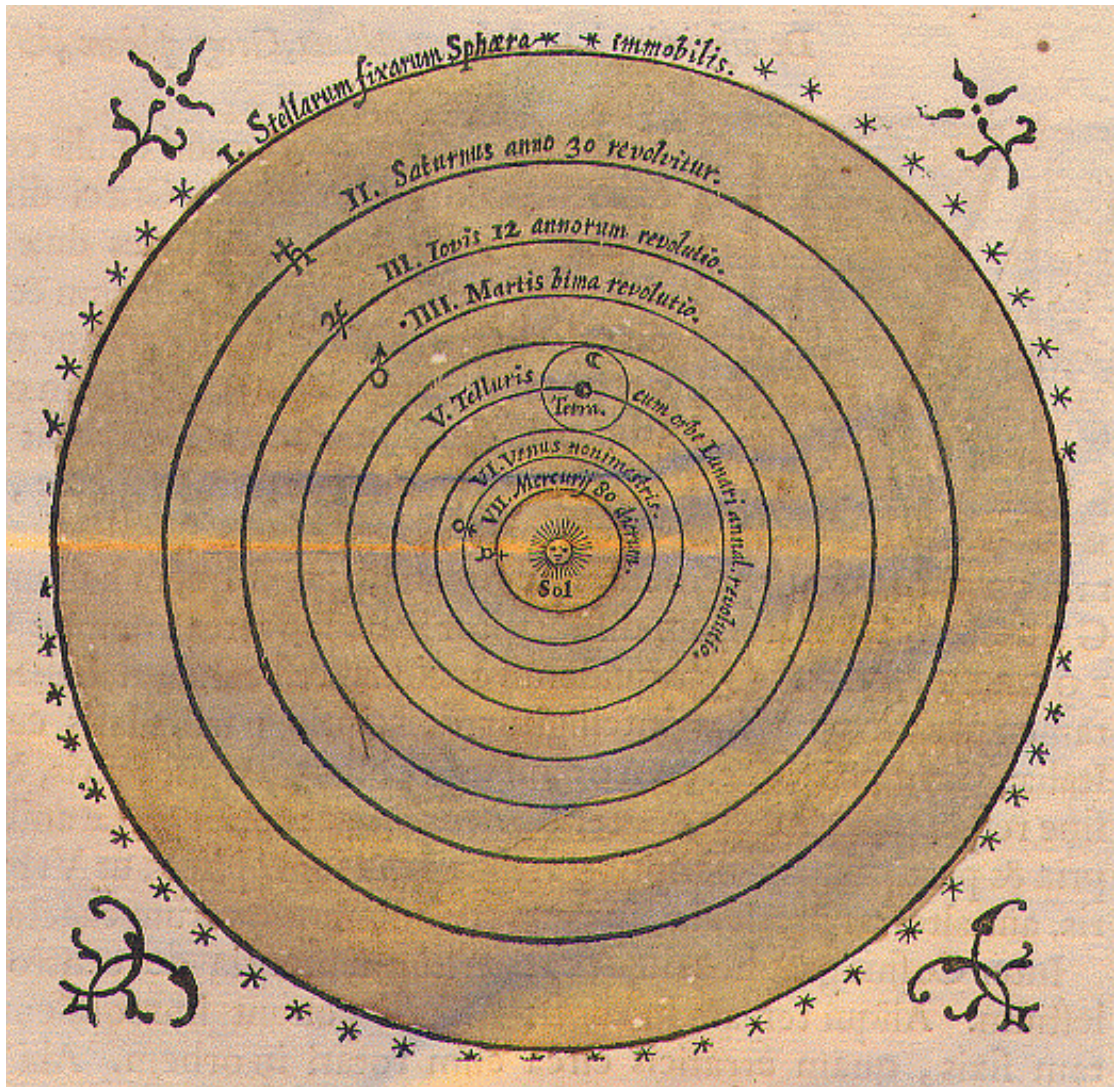


Not-so-simple example: *n*-body problem

- Consider the Earth, moon, and sun—where do they go?
- Solution is trivial for two bodies (e.g., assume one is fixed)
- As soon as n ≥ 3, again get chaotic solutions (no closed form)
- What if we want to simulate entire *galaxies?*



Credit: Governato et al / NASA

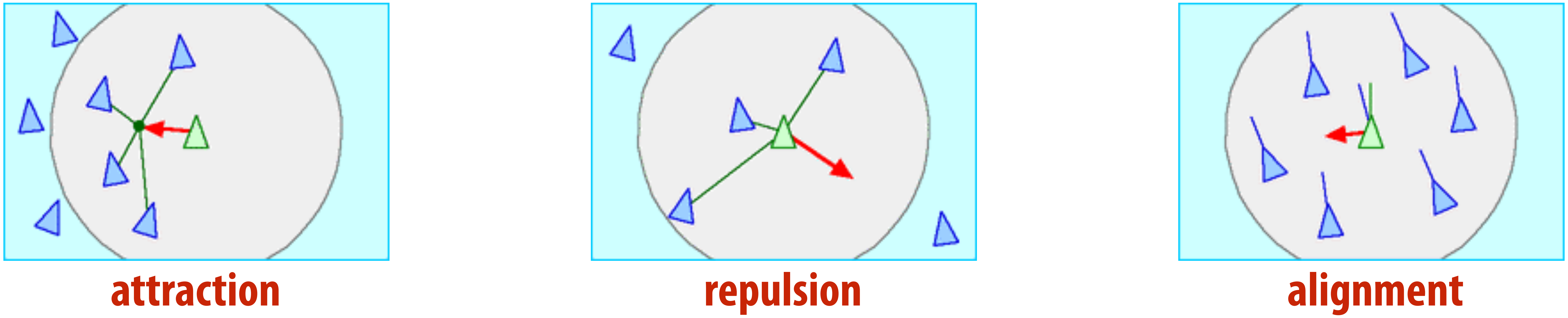


For animation, we *want* to simulate these kinds of phenomena!

Example: flocking

Simulated flocking as an ODE

- Each bird is a particle
- Subject to very simple forces:
  - *attraction* to center of neighbors
  - *repulsion* from individual neighbors
  - *alignment* toward average trajectory of neighbors
- Solve large system of ODEs (numerically!)
- Emergent complex behavior (also seen in fish, bees, ...)



Credit: Craig Reynolds (see http://www.red3d.com/cwr/boids/)

Particle systems

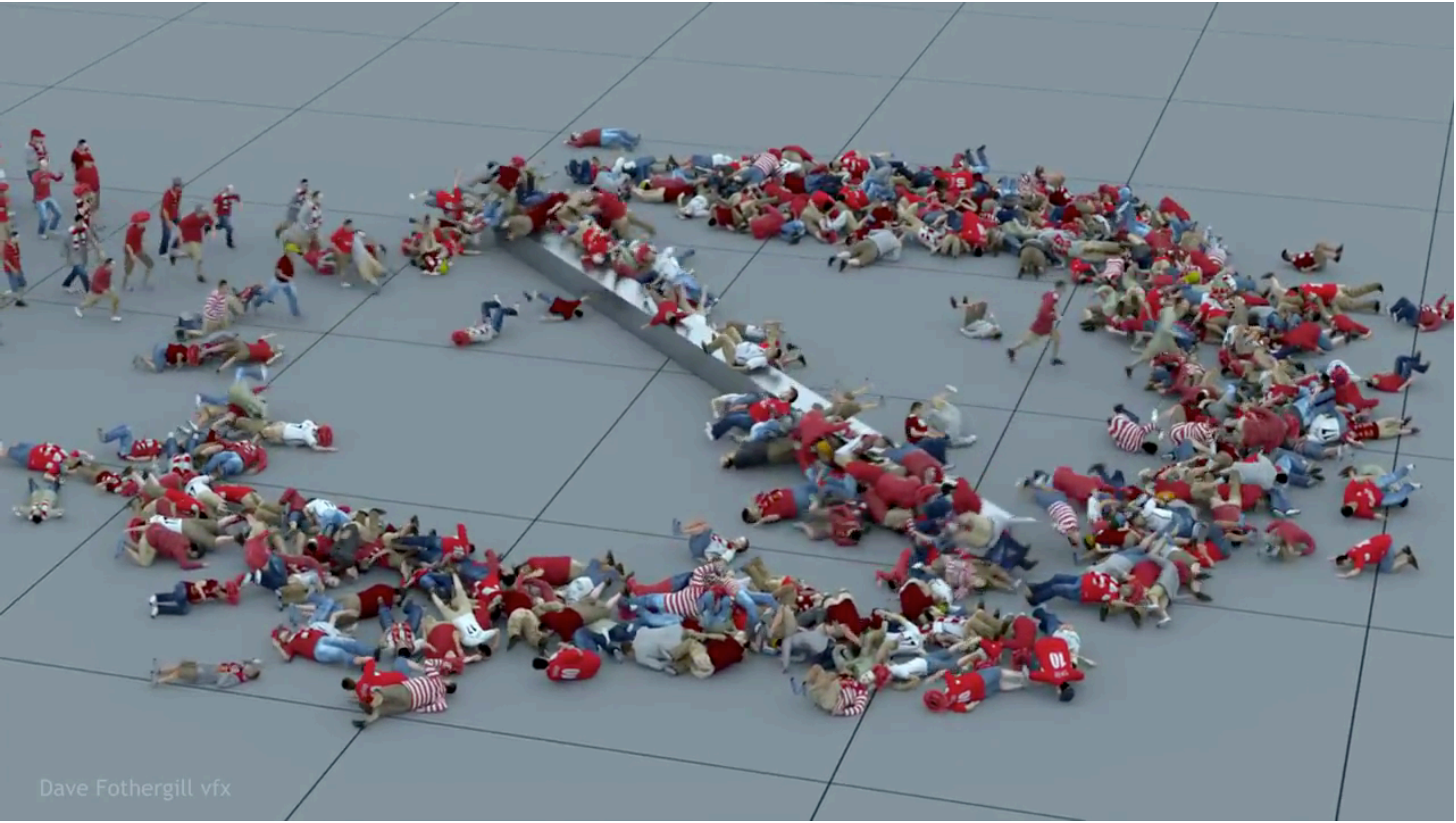
- Model phenomena as large collection of particles
- Each particle has a behavior described by (physical or *non*-physical) forces
- Extremely common in graphics/games
  - easy to understand
  - simple equation for each particle
  - easy to scale up/down

Credit: LMHPoly https://www.youtube.com/watch?v=MYU1vBcUkfw

Example: crowds

Where are the bottlenecks in a building plan?

Example: crowds + “rock” dynamics

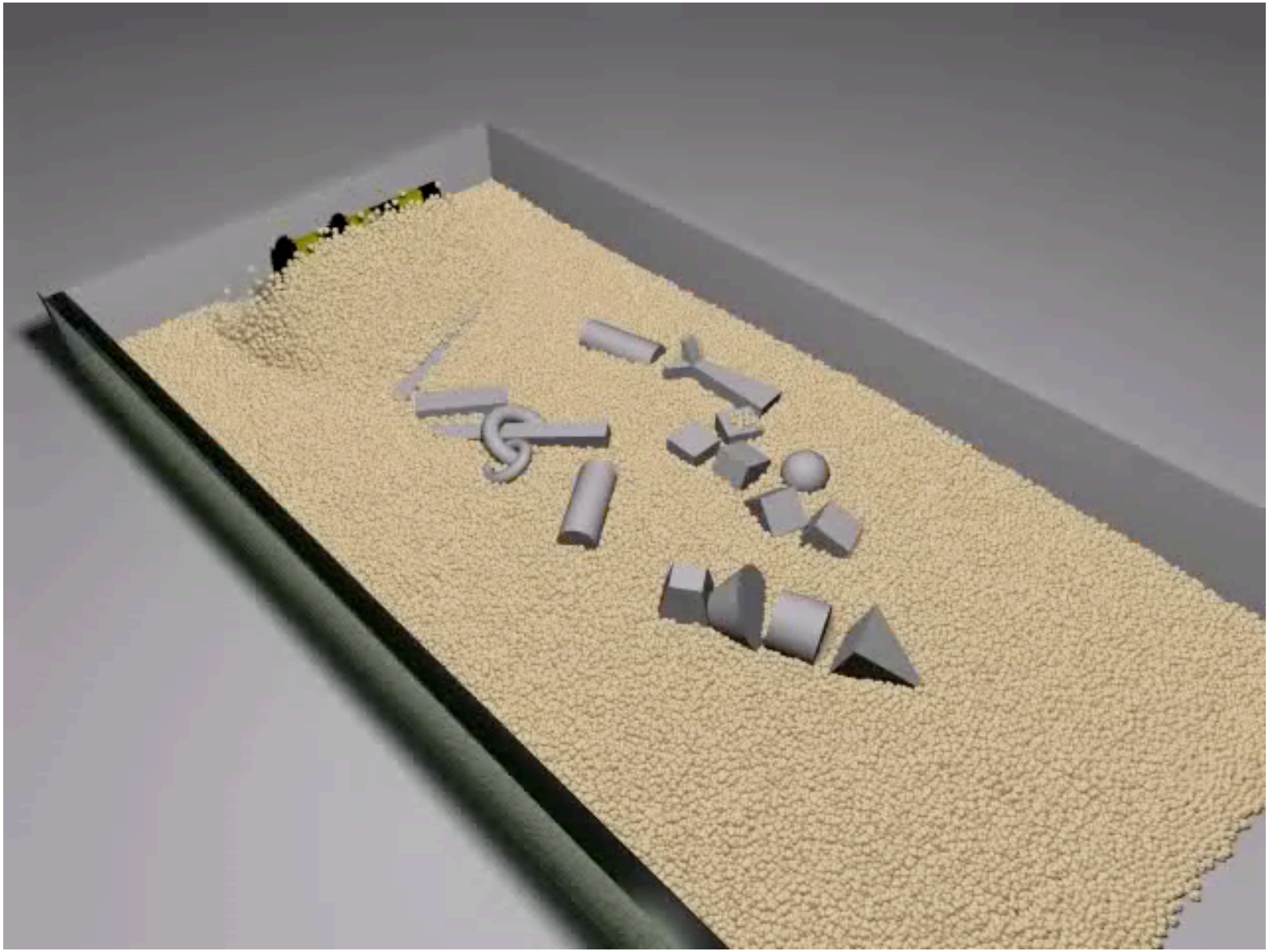


Example: particle-based fluids

Macklin and Müller, Position Based Fluids

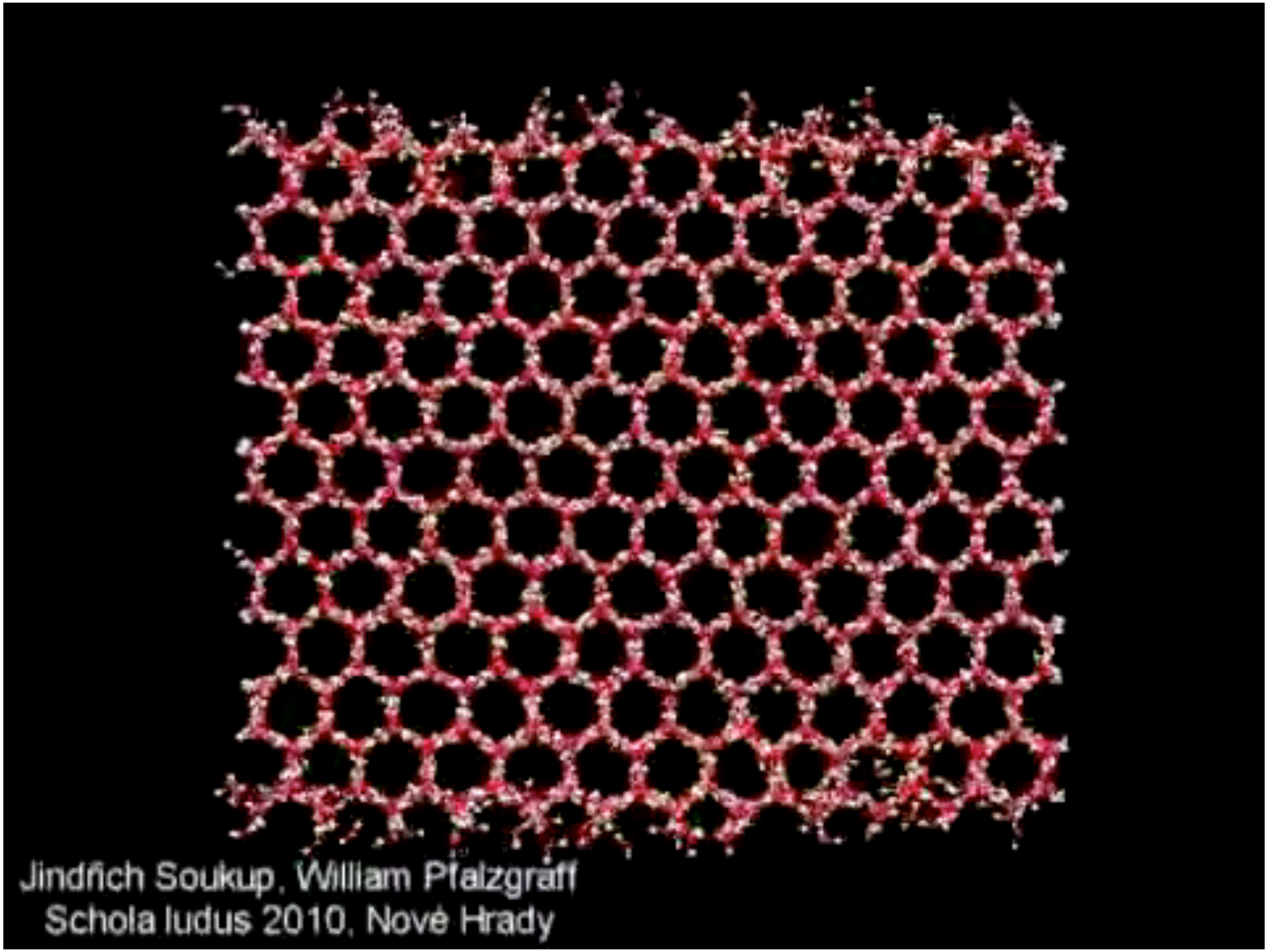
(Fluid: particles or continuum?)

Example: granular materials



Bell et al, “Particle-Based Simulation of Granular Materials”

Example: molecular dynamics



Jindřich Soukup, William Pfalzgraff
Schola ludus 2010, Nové Hrady

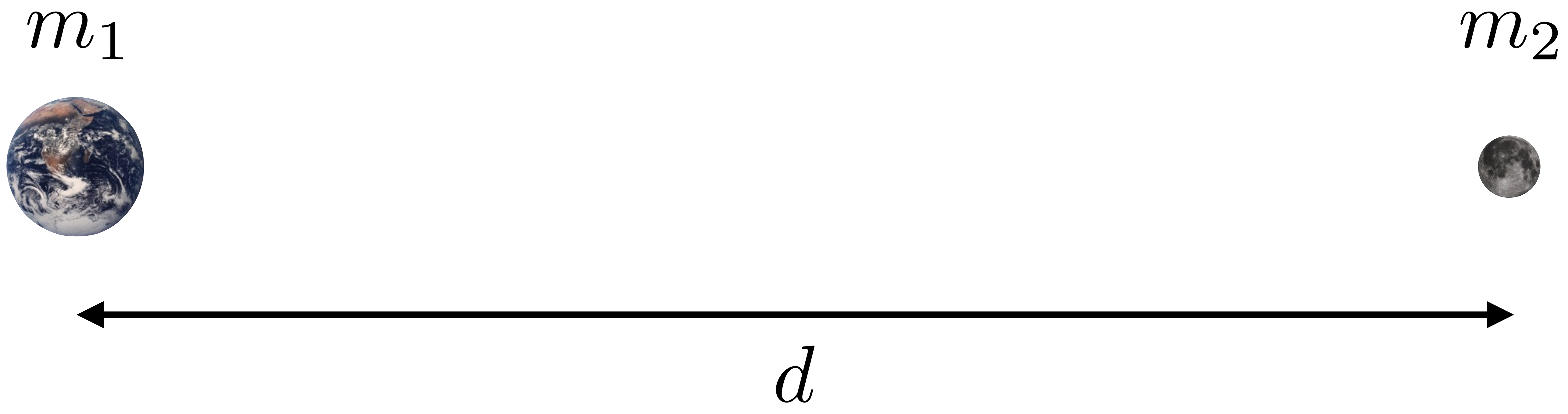
(model of melting ice crystal)

Gravitational attraction

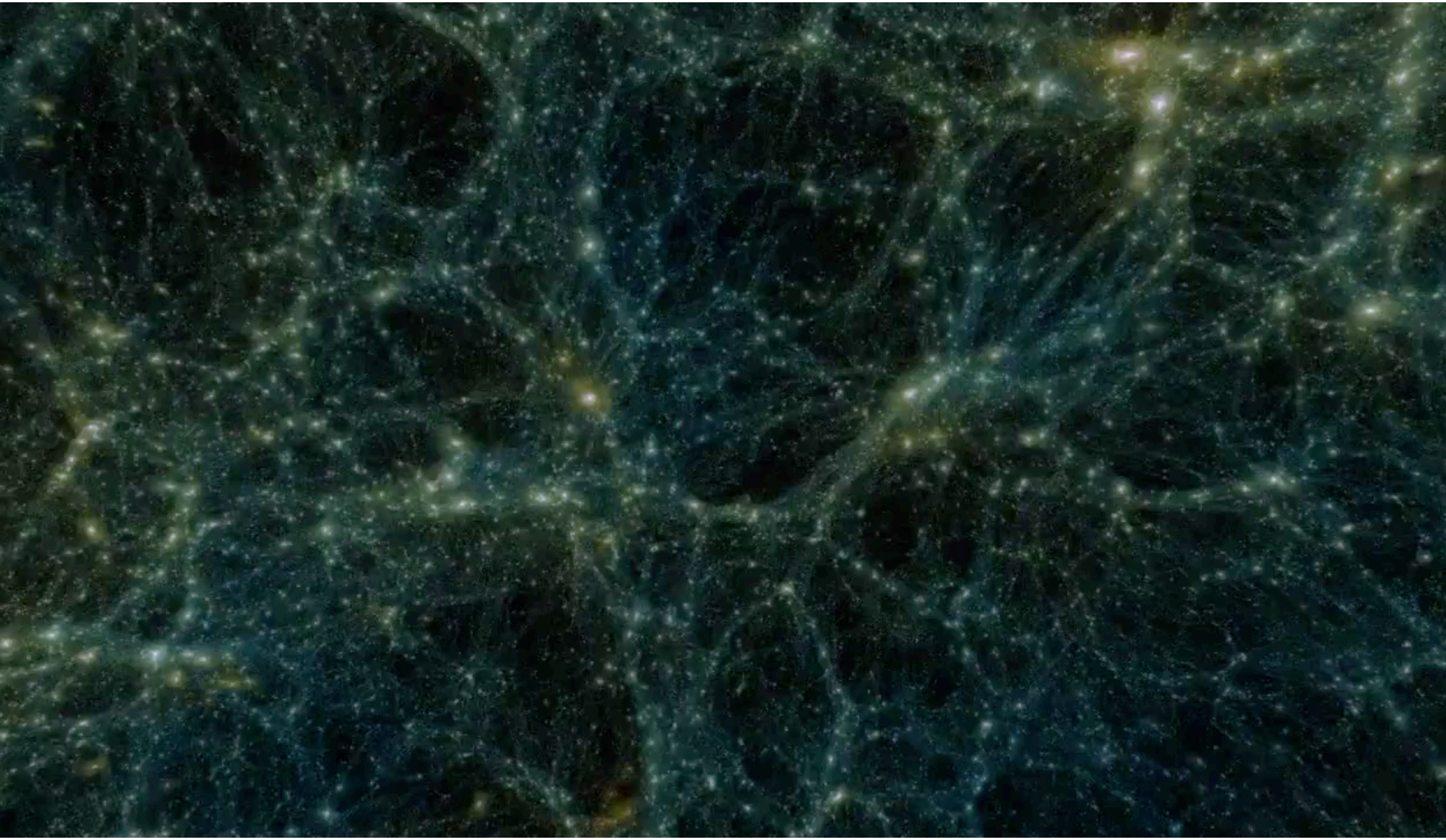
- **Newton's universal law of gravitation**
  - **Gravitational pull between particles**

$$F_g = G\frac{m_1 m_2}{d^2}$$

$$G = 6.67428 \times 10^{-11}\,\mathrm{Nm^2kg^{-2}}$$



Example: cosmological simulation



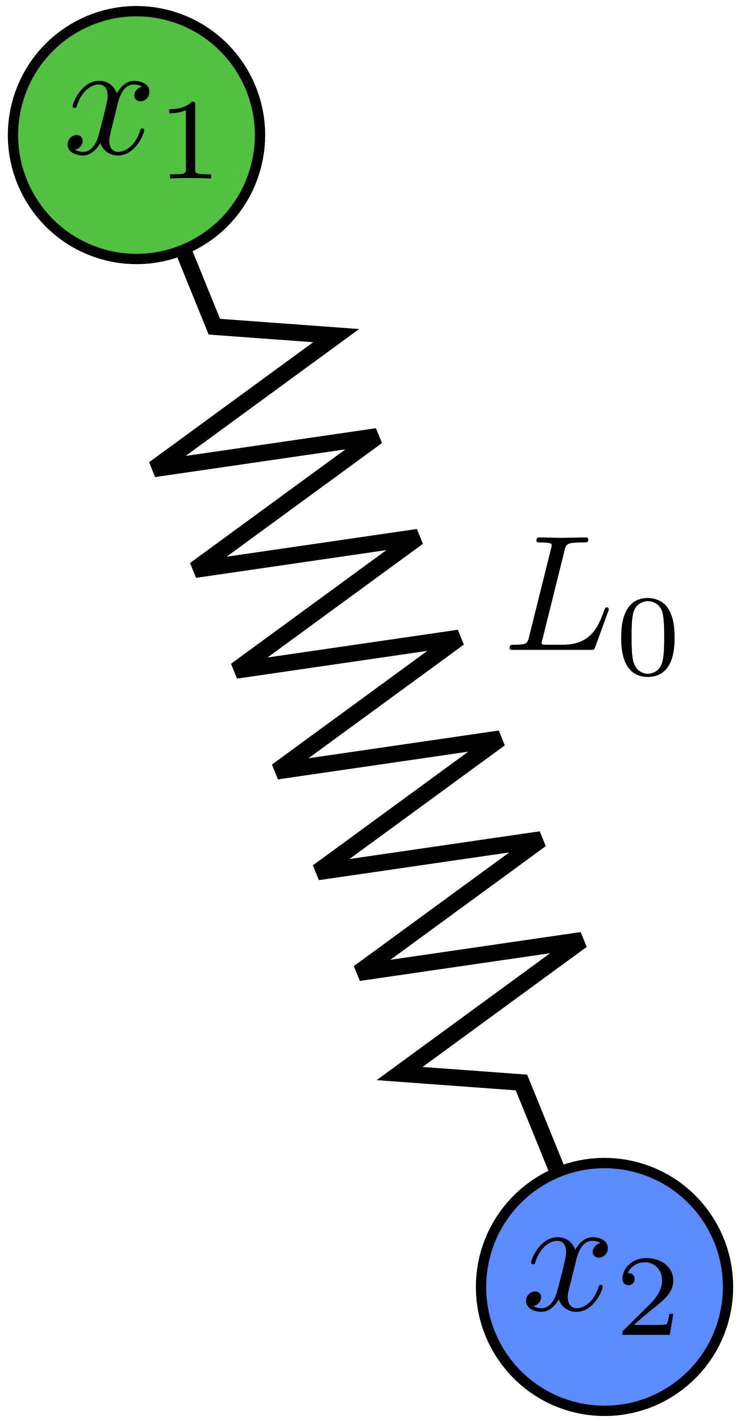
Tomoaki et al - v^2GC simulation of dark matter (~1 *trillion* particles)

Example: mass-spring system

- Connect particles x_1, x_2 by a spring of length L_0
- Potential energy is given by

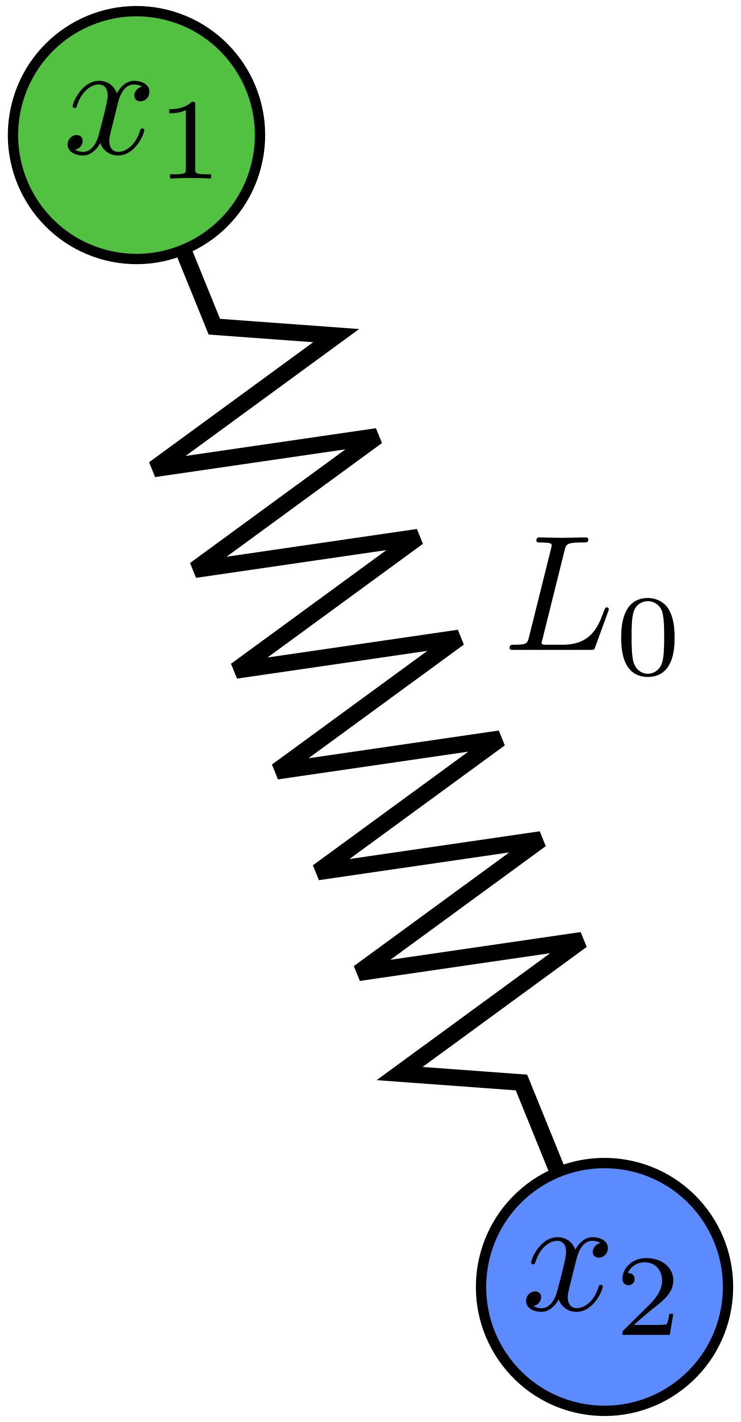
$$U = \frac{1}{2}\underset{\text{stiffness}}{\underset{\uparrow}{k}}(\underset{\text{current length}}{\underset{\uparrow}{L}} - \overset{\text{rest length}}{\overset{\downarrow}{L_0}})^2$$

$$= \frac{1}{2}k(|x_1 - x_2| - L_0)^2$$



- Connect up many springs to describe interesting phenomena
- Extremely common in graphics/games
  - easy to understand
  - simple equation for each particle

Non-zero length spring



- Spring with non-zero rest length
  - Below: direct specification of force on x1 due to spring)

$$f_{\mathbf{x_1}} = k(|\mathbf{x_2} - \mathbf{x_1}| - L_0)$$

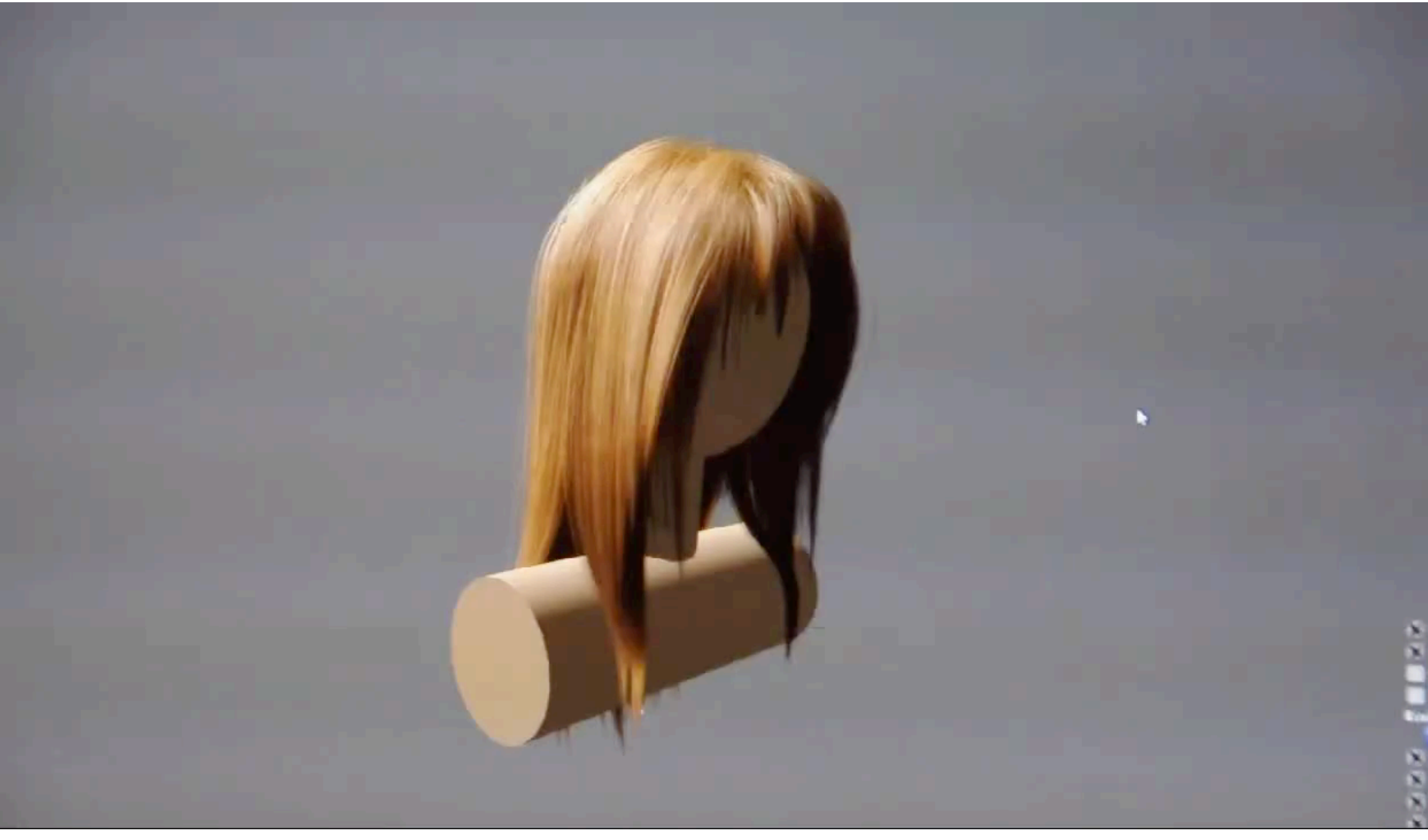
Problem: oscillates forever…

How might we add internal dampening?

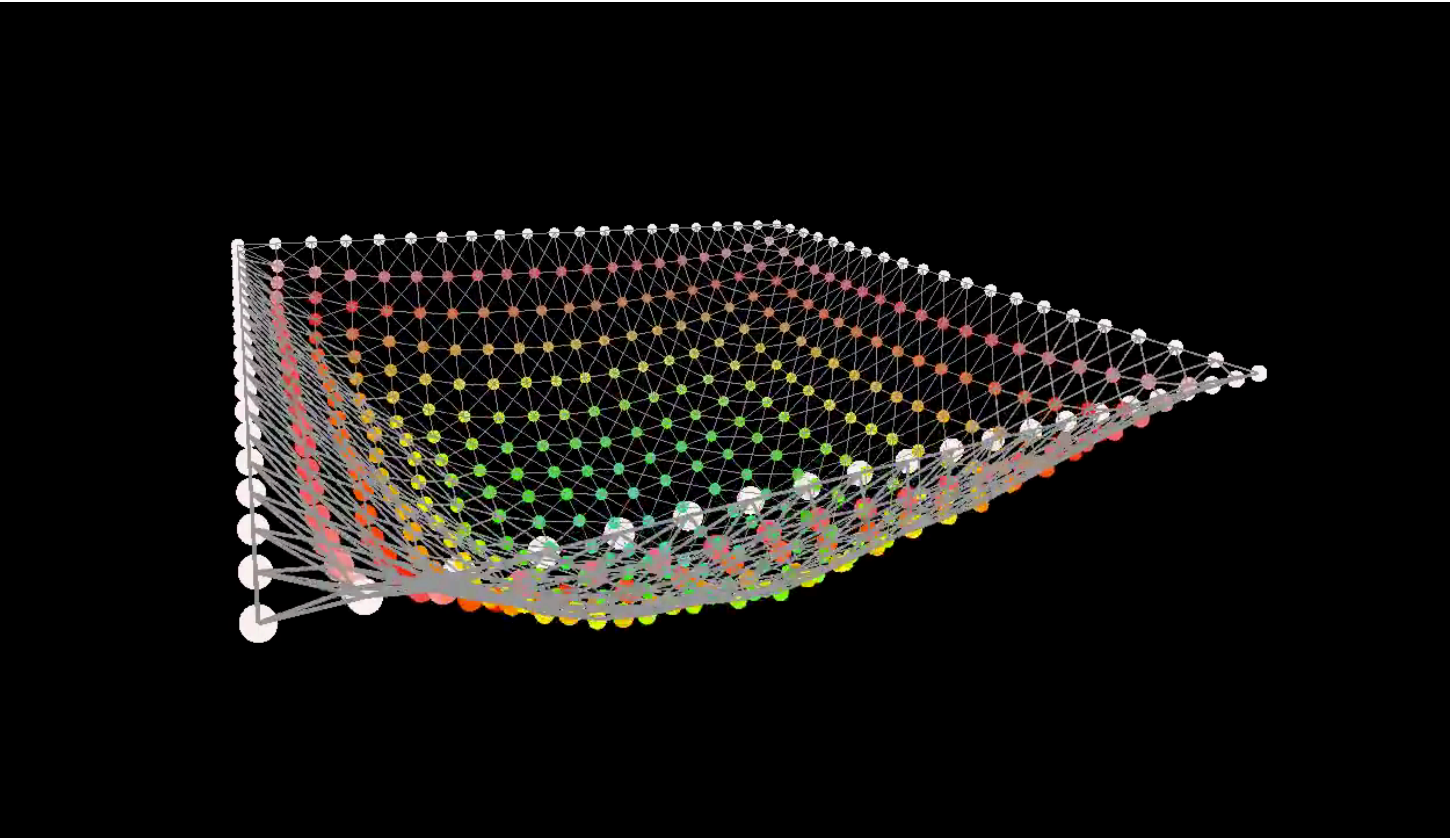
Example: mass-spring rope

Credit: Elizabeth Labelle, https://youtu.be/Co8enp8CH34

Example: hair

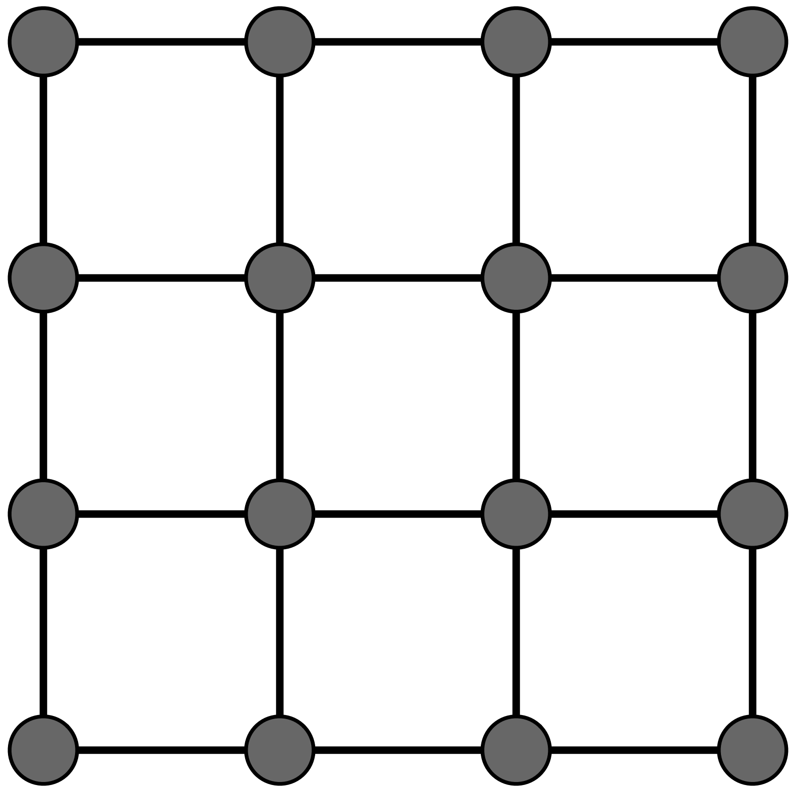


Example: mass-spring system

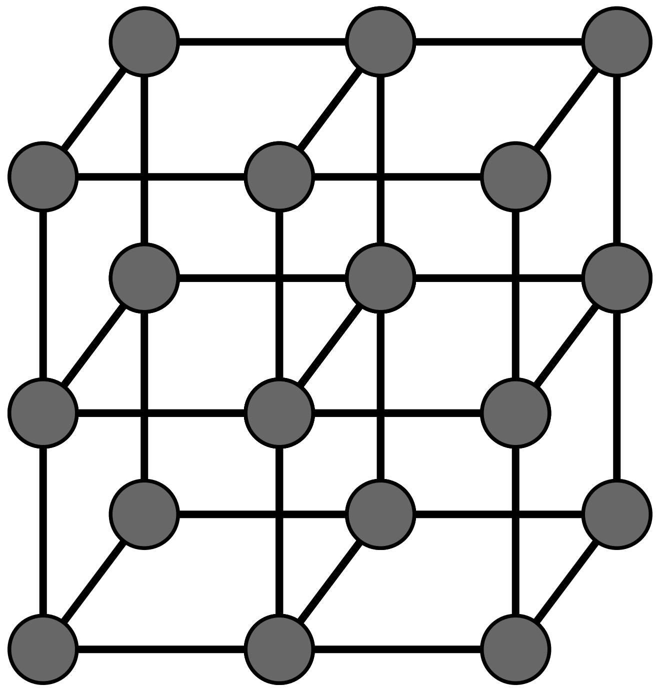


Example structures from springs

- **Sheets**

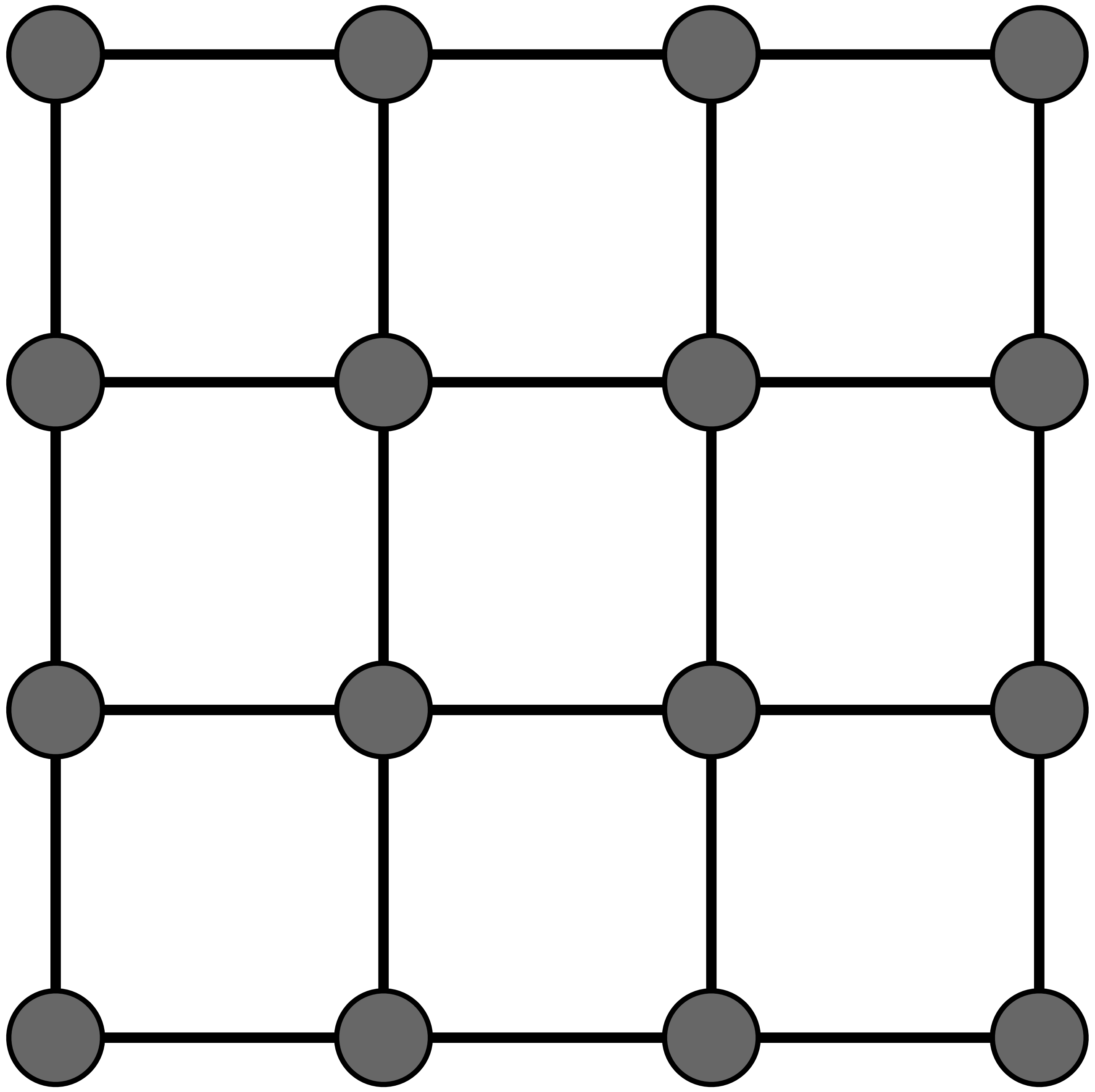


- **Blocks**



Structures from springs

- **Behavior is determined by structure linkages**

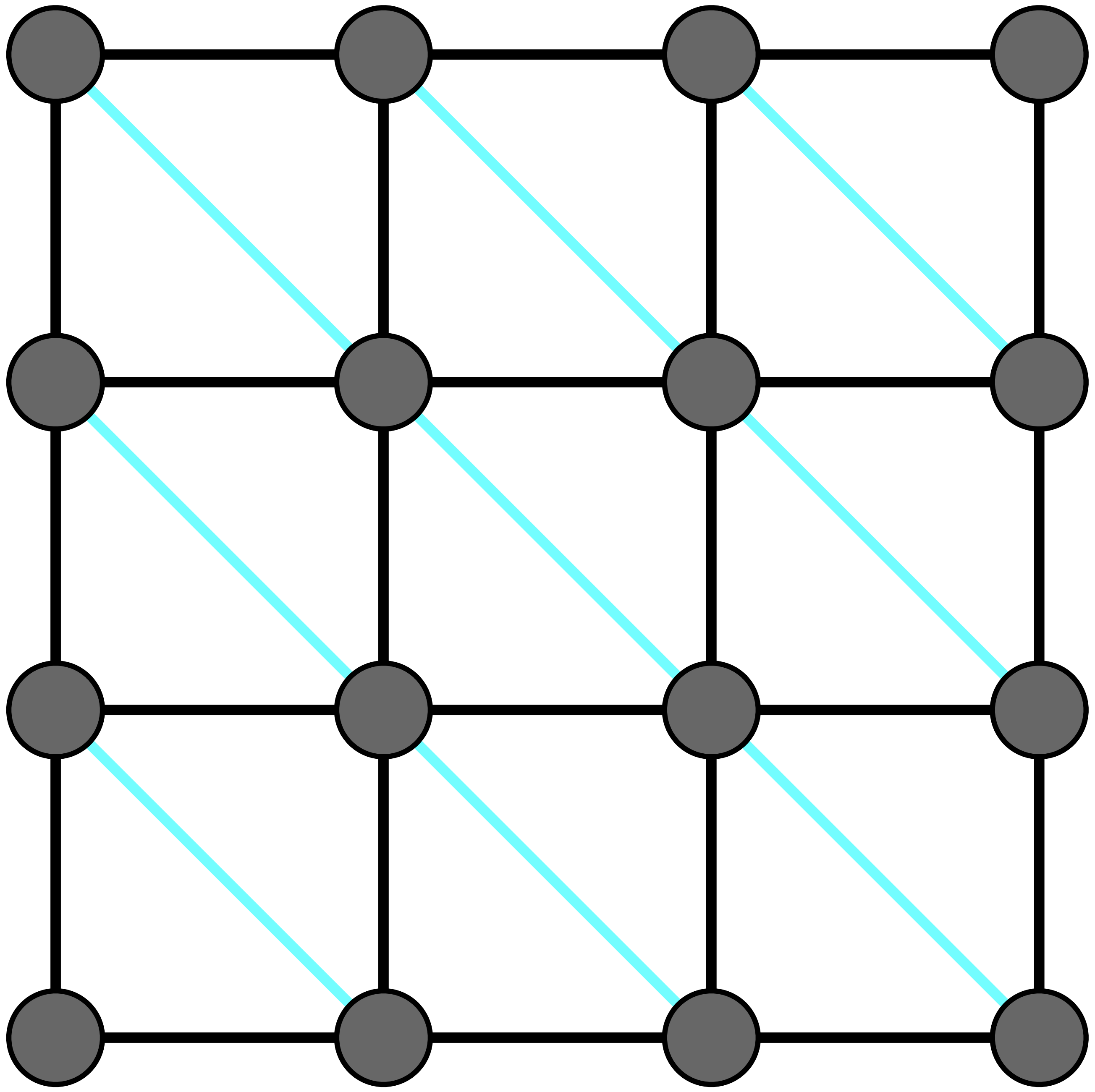


This structure will not resist shearing

It will also not resist out-of-plane bending.

Structures from springs

- **Behavior is determined by structure linkages**

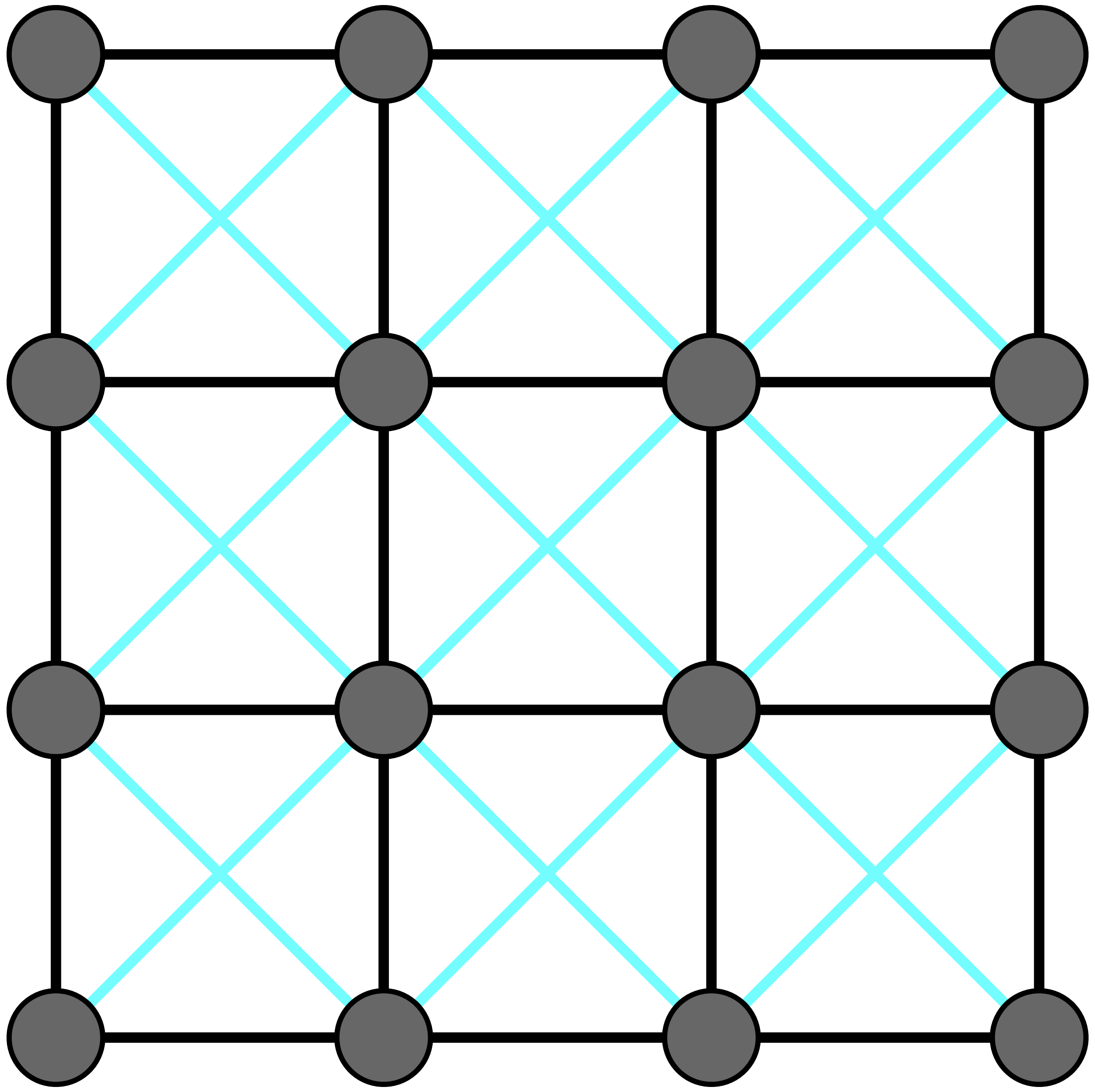


This structure will resist shearing but has anisotropic bias

It will not resist out-of-plane bending.

Structures from springs

- **Behavior is determined by structure linkages**

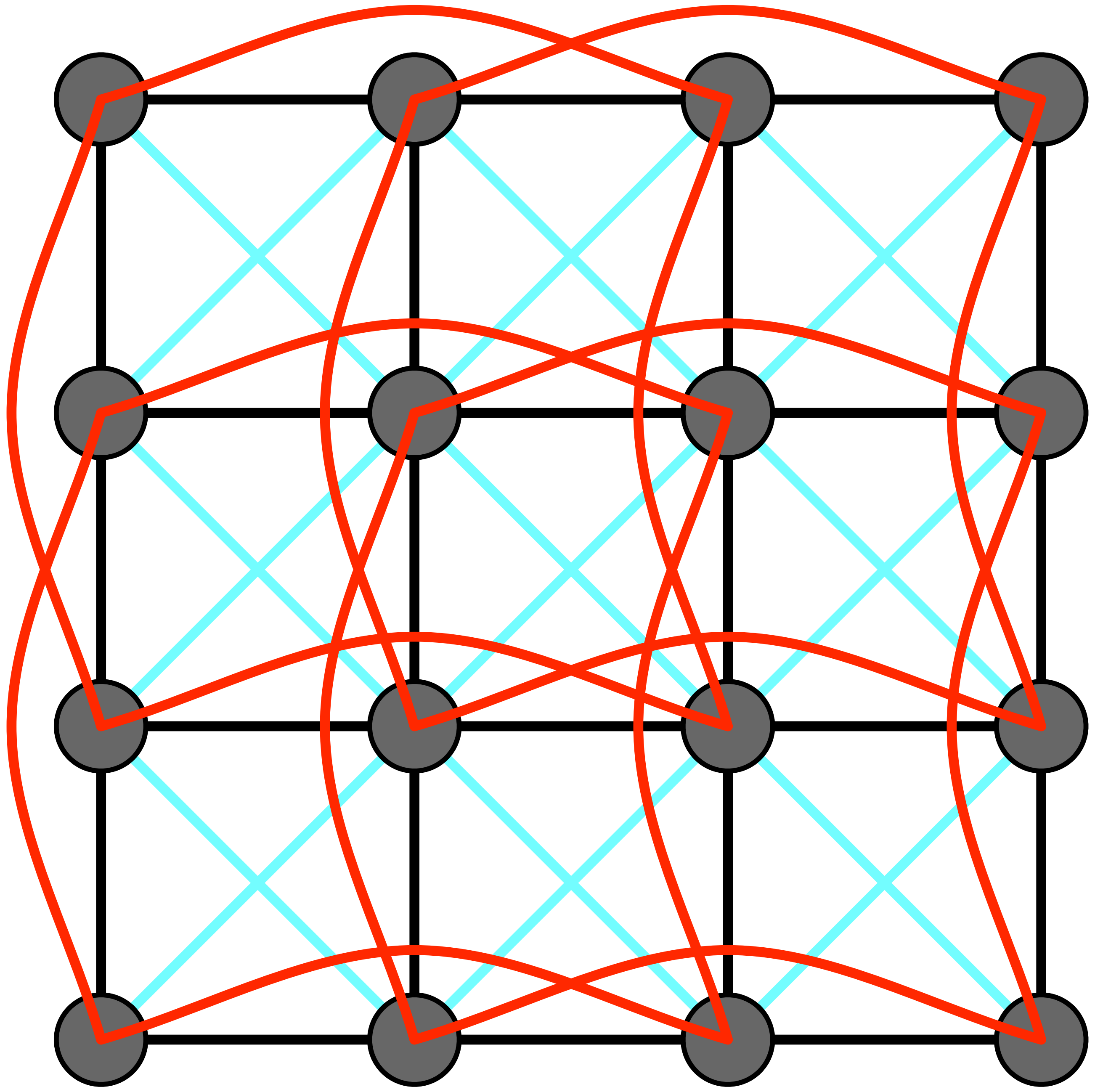


This structure will resist shearing. Less directional bias.

But will not resist out-of-plane bending...

Structures from springs

- Behavior is determined by structure linkages

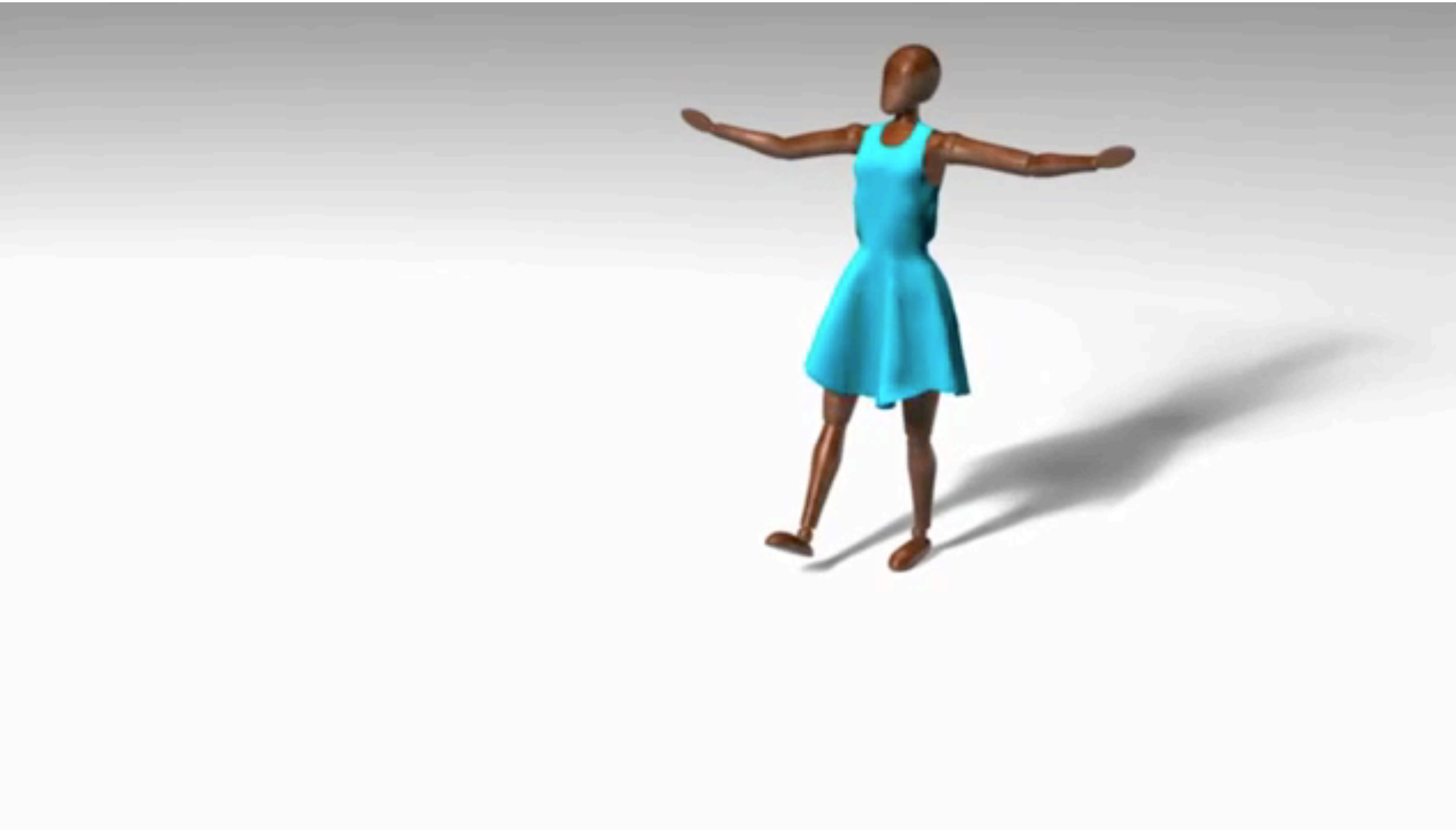


This structure not resist shearing.
Less directional bias.

This structure will resist out-of-plane bending.

In general, red springs should be weaker

Example: mass spring + character anim



How do we solve these systems numerically?

Numerical integration

- **Key idea: replace *derivatives* with *differences***
- **In ODE, only need to worry about derivative in *time***
- **Replace time-continuous function q(t) with samples q_k in time**

$$\frac{d}{dt}q(t) = f(q(t))$$

$$\Downarrow$$

new configuration
(unknown—want to solve for this!)

current configuration
(known)

$$\frac{q_{k+1} - q_k}{\tau} = f(q)$$

"time step," i.e., interval of time between q_k and q_{k+1}

Wait... where do we evaluate the velocity function? At the new or old configuration?

Forward Euler

- **Simplest scheme: evaluate velocity at current configuration**
- **New configuration can then be written *explicitly* in terms of known data:**

$$\overset{\substack{\text{new configuration}\\ \searrow}}{q_{k+1}} = \overset{\substack{\text{current configuration}\\ \downarrow}}{q_k} + \tau \overset{\substack{\text{velocity at current time}\\ \downarrow}}{f}(q_k)$$

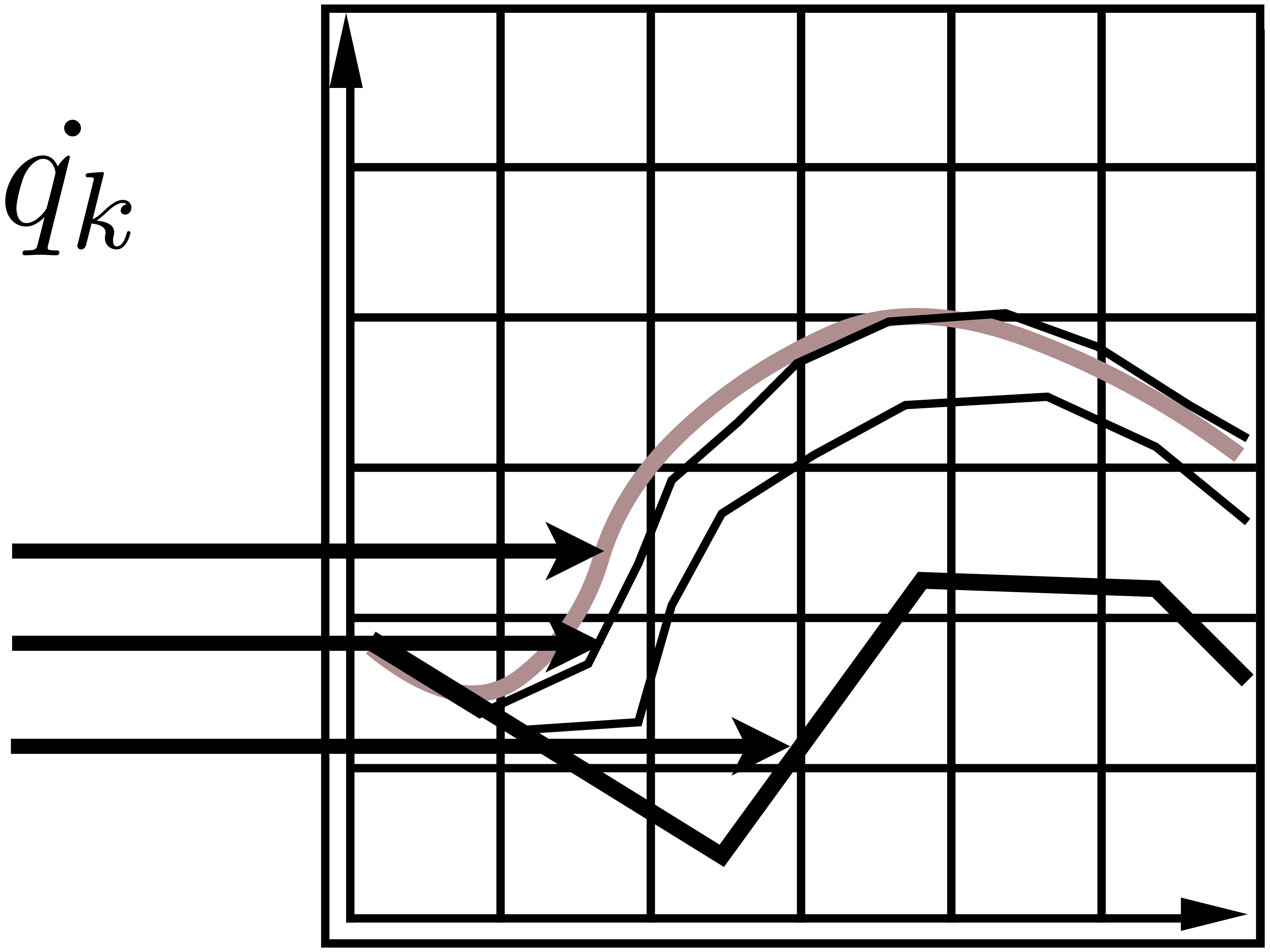
- **Very intuitive: walk a tiny bit in the direction of the velocity**
- **Problems: poor accuracy and not very *stable***

Euler's method - error

- **With numerical integration, errors accumulate**

Example:

$$q_{k+1} = q_k + \tau \dot{q}_k$$



Solution path

Euler estimate with small time step

Euler estimate with large time step

Witkin and Baraff

Problem: instability

$$q_{k+1} = q_k + \tau \dot{q}_k$$

- Very intuitive: walk a tiny bit in the direction of the velocity
- Unfortunately, not very *stable, consider a spring…*

When mass is moving inward:

- Force is decreasing
- Each time-step overestimates the velocity change (increases energy)

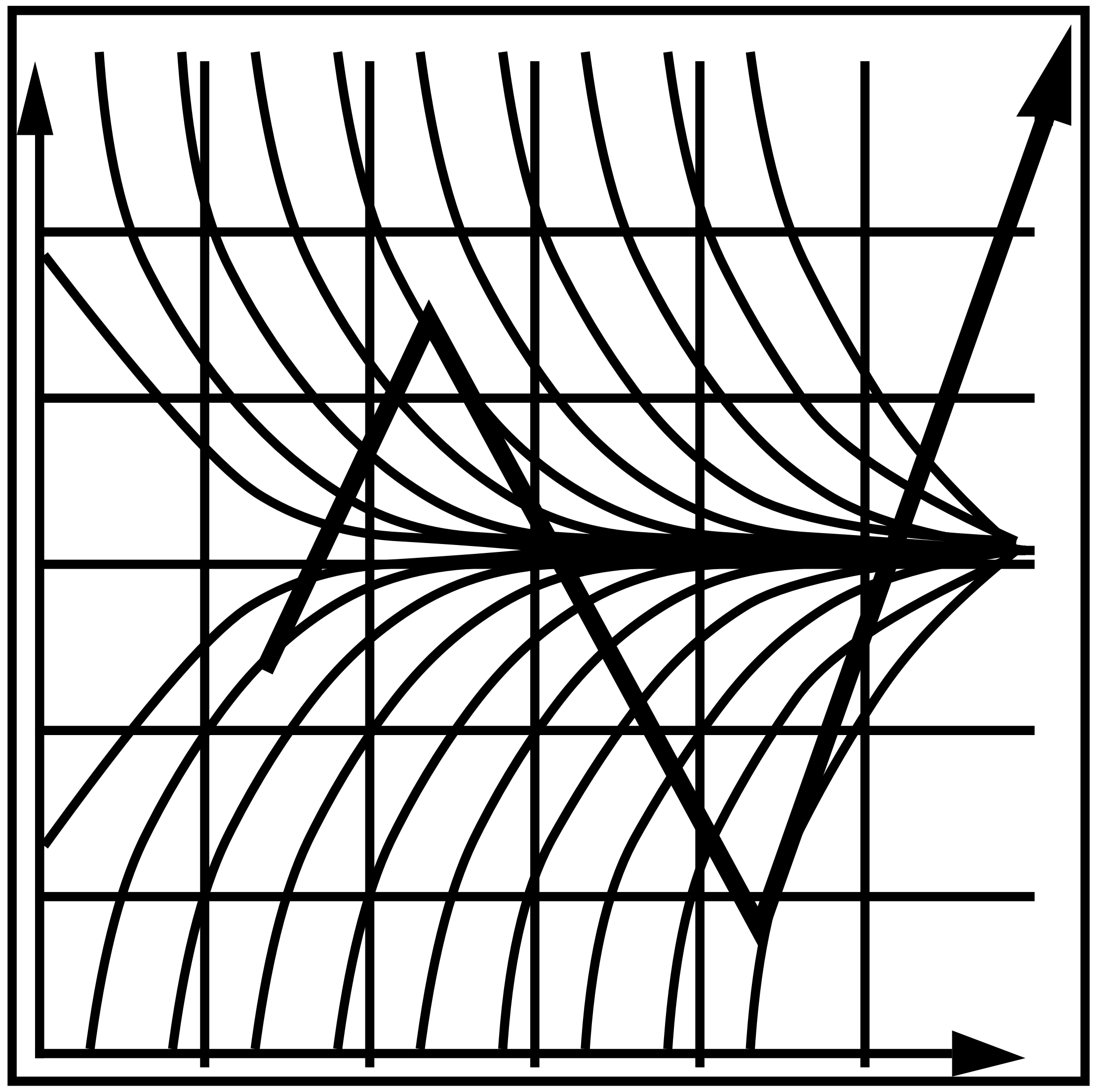
When mass gets to origin

- Has velocity that is too high, now traveling outward

When mass is moving outward

- Force is increasing
- Each time-step underestimates the velocity change (increases energy)

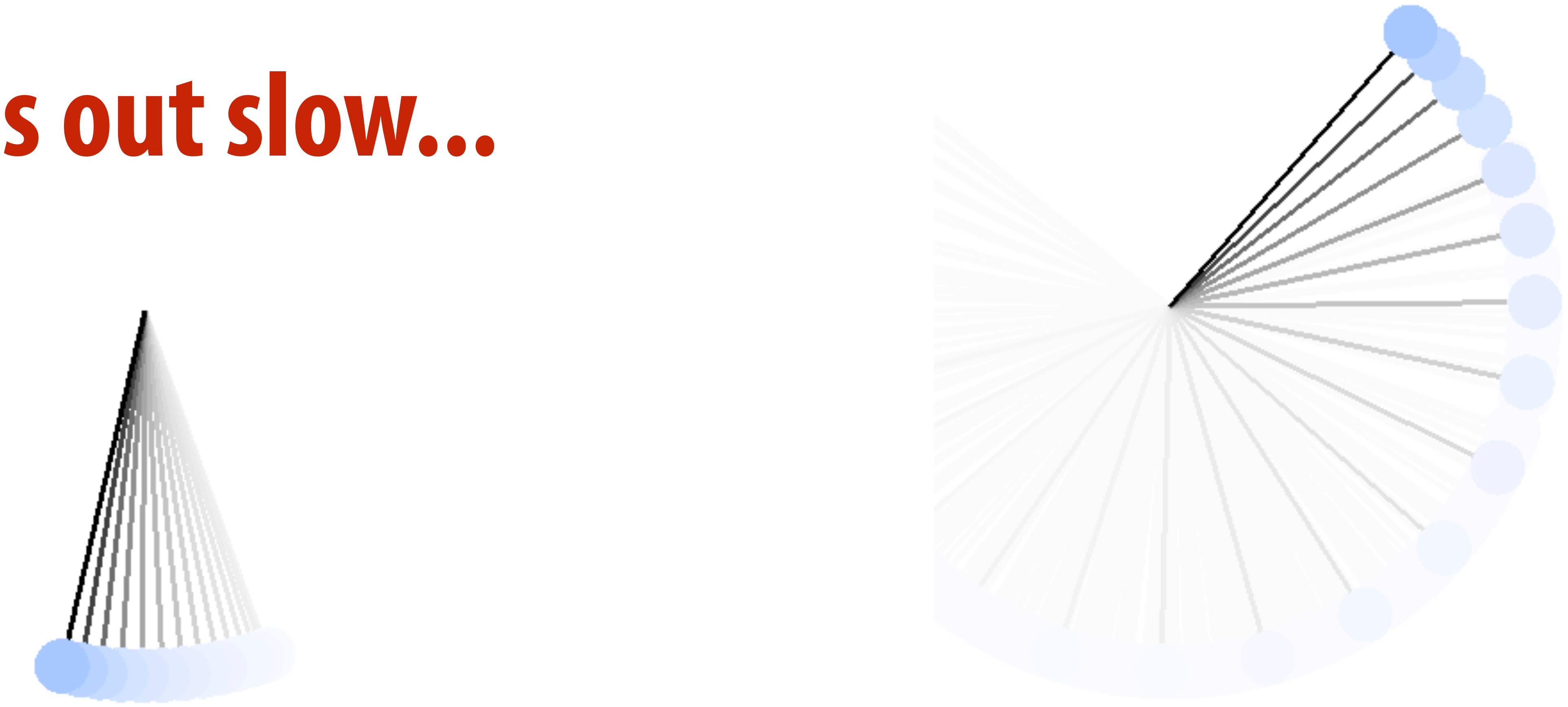
With each motion cycle, mass gains energy exponentially



Another example

- Consider a pendulum…

starts out slow...



Where did all this extra energy come from?

...gradually moves faster & faster!

Forward Euler - stability analysis

- **Let's consider behavior of forward Euler for simple linear ODE:**

$$\dot{u} = -au, \quad a > 0$$

- **Importantly: *u* should *decay* (exact solution is *u*(*t*)=e^{-at})**
- **Forward Euler approximation is**

$$\begin{aligned} u_{k+1} &= u_k - \tau a u_k \\ &= (1 - \tau a)u_k \end{aligned}$$

- **Which means after n steps, we have**

$$u_n = (1 - \tau a)^n u_0$$

- **Decays only if $|1-\tau a| < 1$, or equivalently, if $\tau < 2/a$**
- **In practice: need *very small* time steps if a is large ("stiff system")**

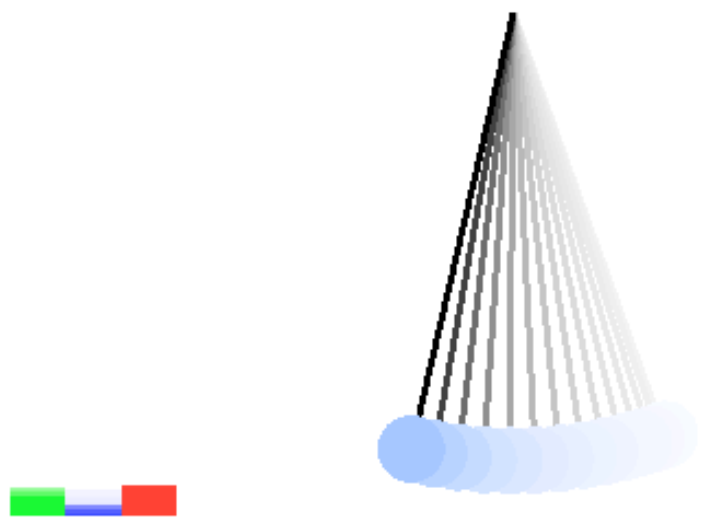
Backward Euler

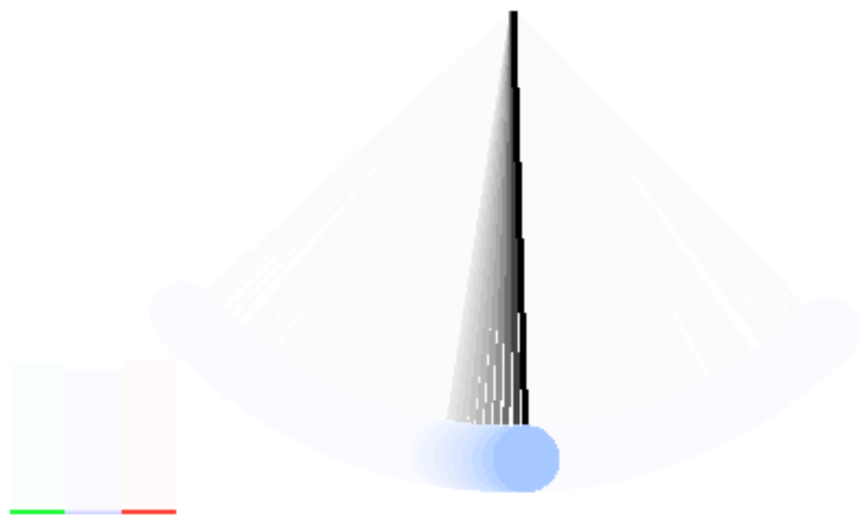
- Let's try something else: evaluate velocity at *next* configuration
- New configuration is then *implicit*, and we must solve for it:

$$\overset{\substack{\text{new configuration}\\ \searrow}}{q_{k+1}} = \overset{\substack{\text{current configuration}\\ \downarrow}}{q_k} + \tau \overset{\substack{\text{velocity at next time}\\ \downarrow}}{f}(q_{k+1})$$

- Harder to solve, since in general *f* can be very nonlinear!
- Pendulum is now stable... perhaps *too* stable?

starts out slow...





Where did all the energy go?

...and eventually stops moving completely.

Backward Euler - stability analysis

- **Again consider a simple linear ODE:**

$$\dot{u} = -au, \quad a > 0$$

- **Remember: *u* should *decay* (exact solution is *u*(*t*)=e^{-at})**
- **Backward Euler approximation is**

$$(u_{k+1} - u_k)/\tau = -au_{k+1}$$

$$\iff \quad u_{k+1} = \tfrac{1}{1+\tau a} u_k$$

- **Which means after n steps, we have**

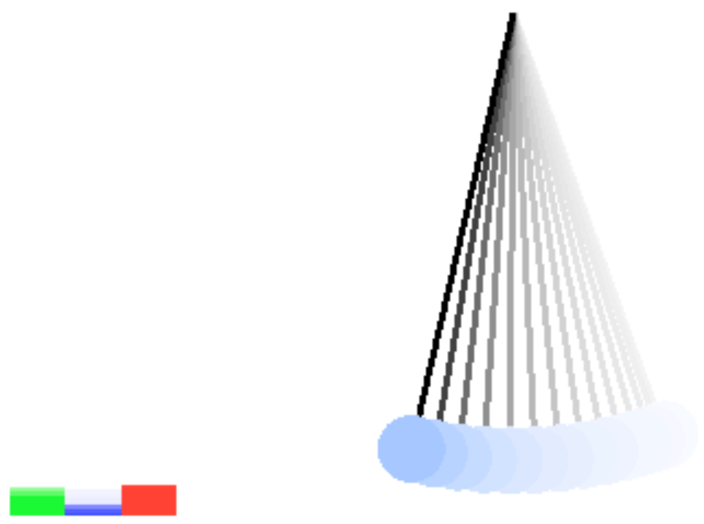
$$u_n = \left(\tfrac{1}{1+\tau a}\right)^n u_0$$

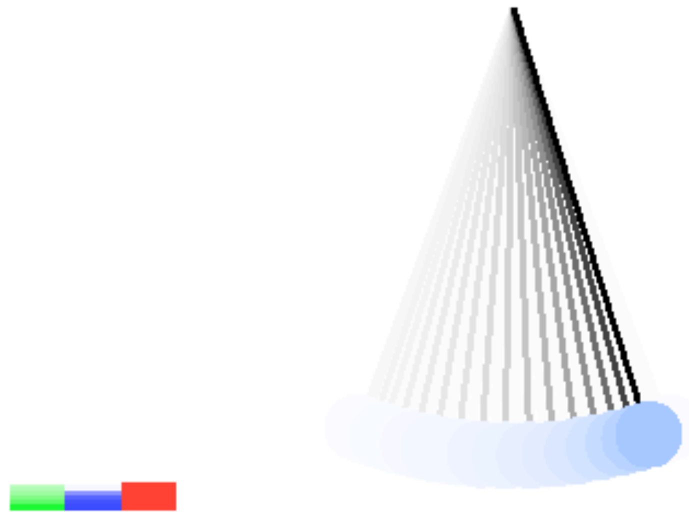
- **Decays if |1+τa| > 1, which is always true!**
- **⇒Backward Euler is *unconditionally stable* for linear ODEs**

Symplectic Euler

- **Backward Euler was stable, but we also saw (empirically) that it exhibits *numerical damping* (damping not found in original eqn.)**
- **Nice alternative is symplectic Euler**
  - **update velocity using current configuration**
  - **update configuration using *new* velocity**
- **Easy to implement; used often in practice**
- **Pendulum now conserves energy *almost exactly*, forever:**

starts out slow...



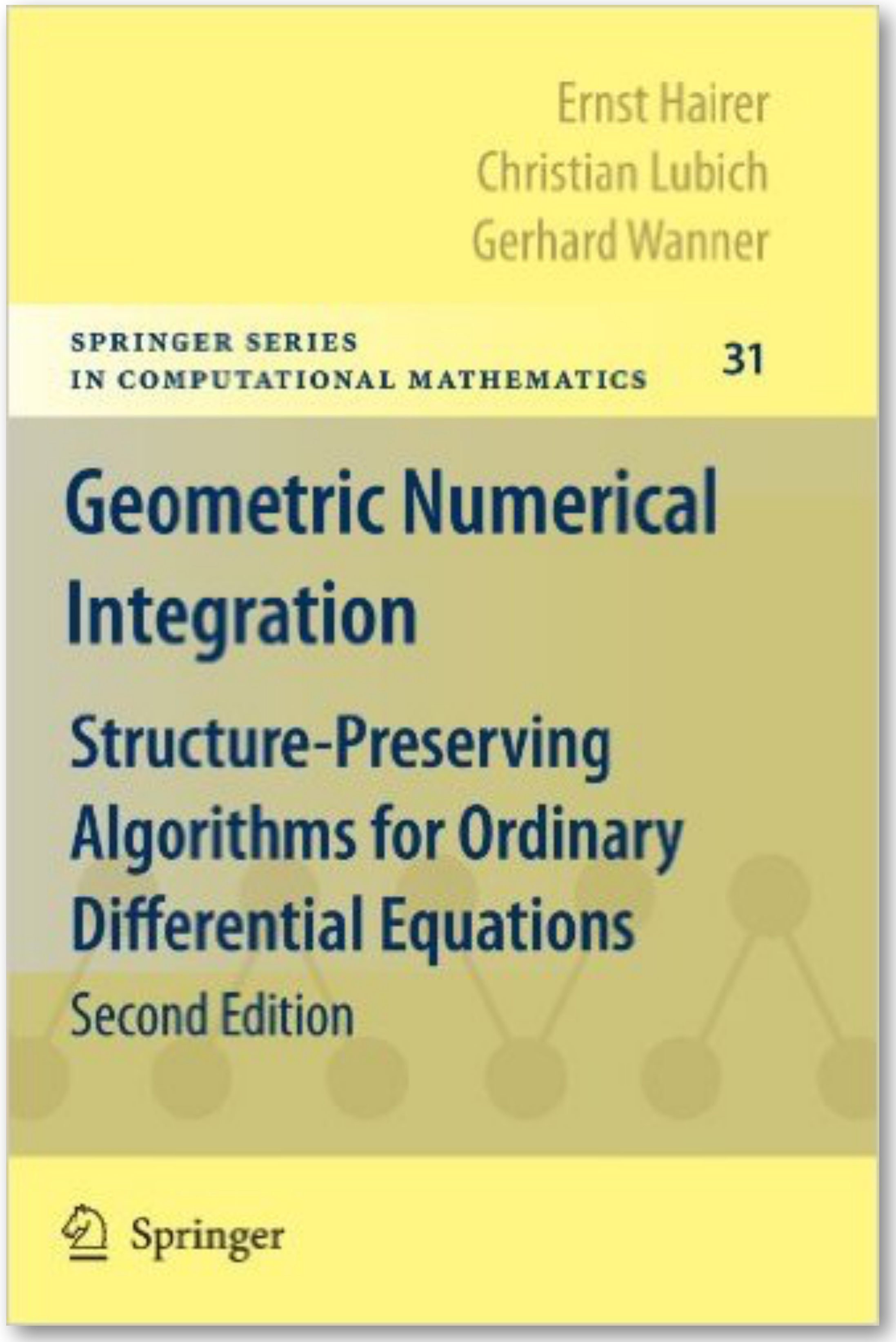


...and keeps on ticking.

(Proof? The analysis is not quite as easy...)

Numerical integrators

- Barely scratched the surface
- *Many* different integrators
- Why? Because many notions of “good”:
  - stability
  - accuracy
  - consistency/convergence
  - conservation, symmetry, …
  - computational efficiency (!)
- No one “best” integrator—*pick the right tool for the job!*
- Could do (at least) an entire course on time integration…
- Great book: Hairer, Lubich, Wanner



Ernst Hairer
Christian Lubich
Gerhard Wanner

SPRINGER SERIES
IN COMPUTATIONAL MATHEMATICS 31

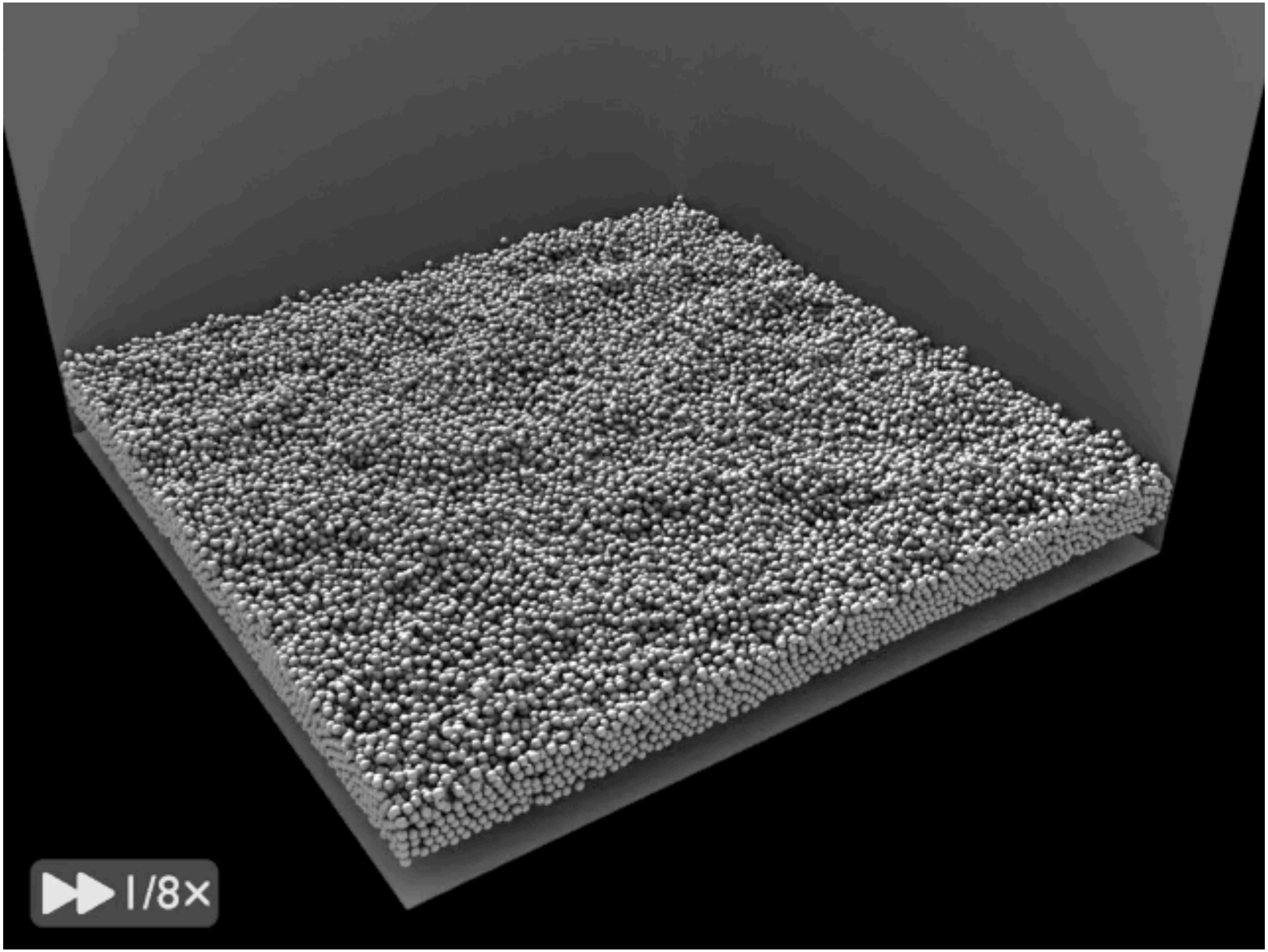
Geometric Numerical Integration

Structure-Preserving Algorithms for Ordinary Differential Equations

Second Edition

Springer

Not covered today: contact mechanics

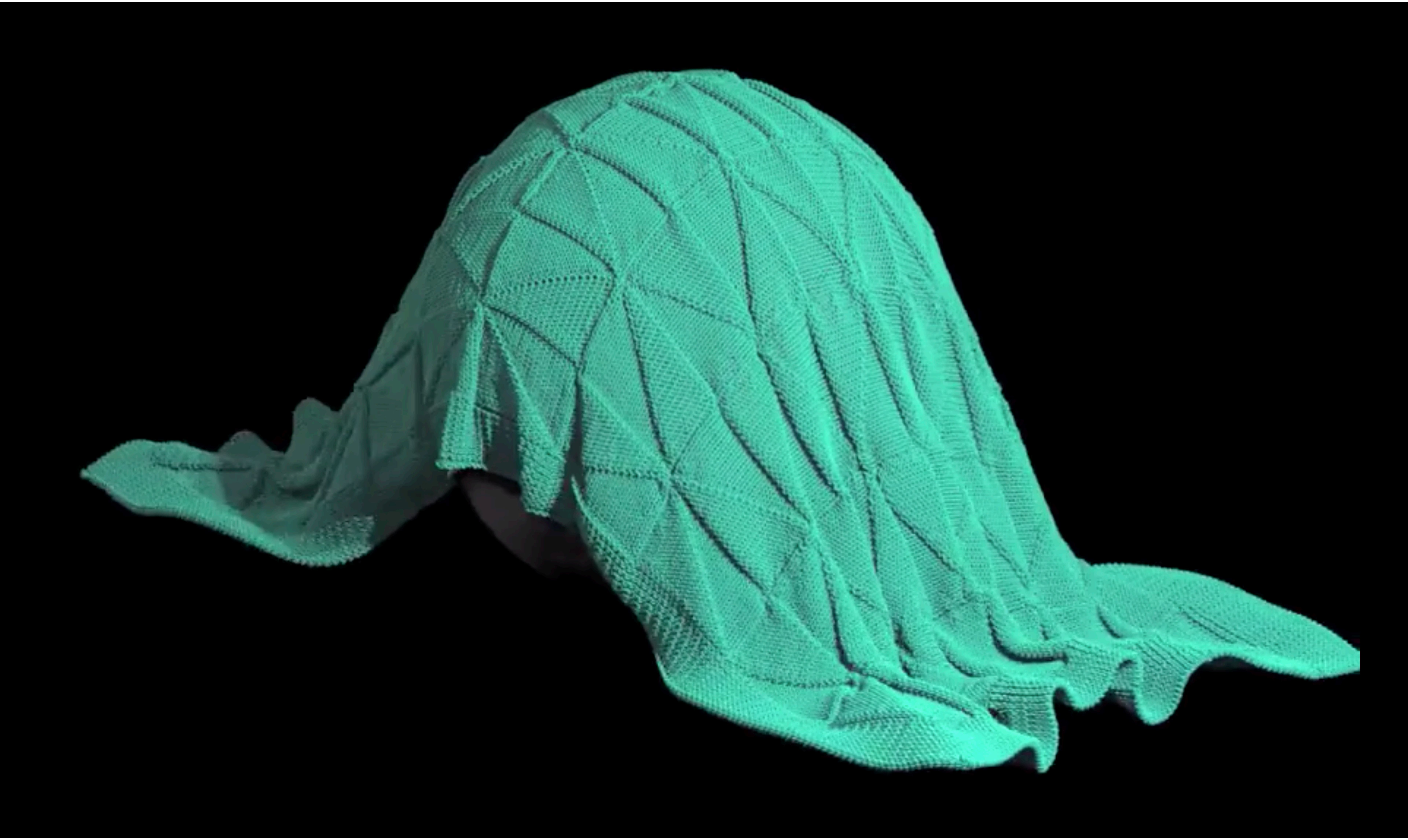


Smith et al, ***"Reflections on Simultaneous Impact"***

Not covered today: contact mechanics

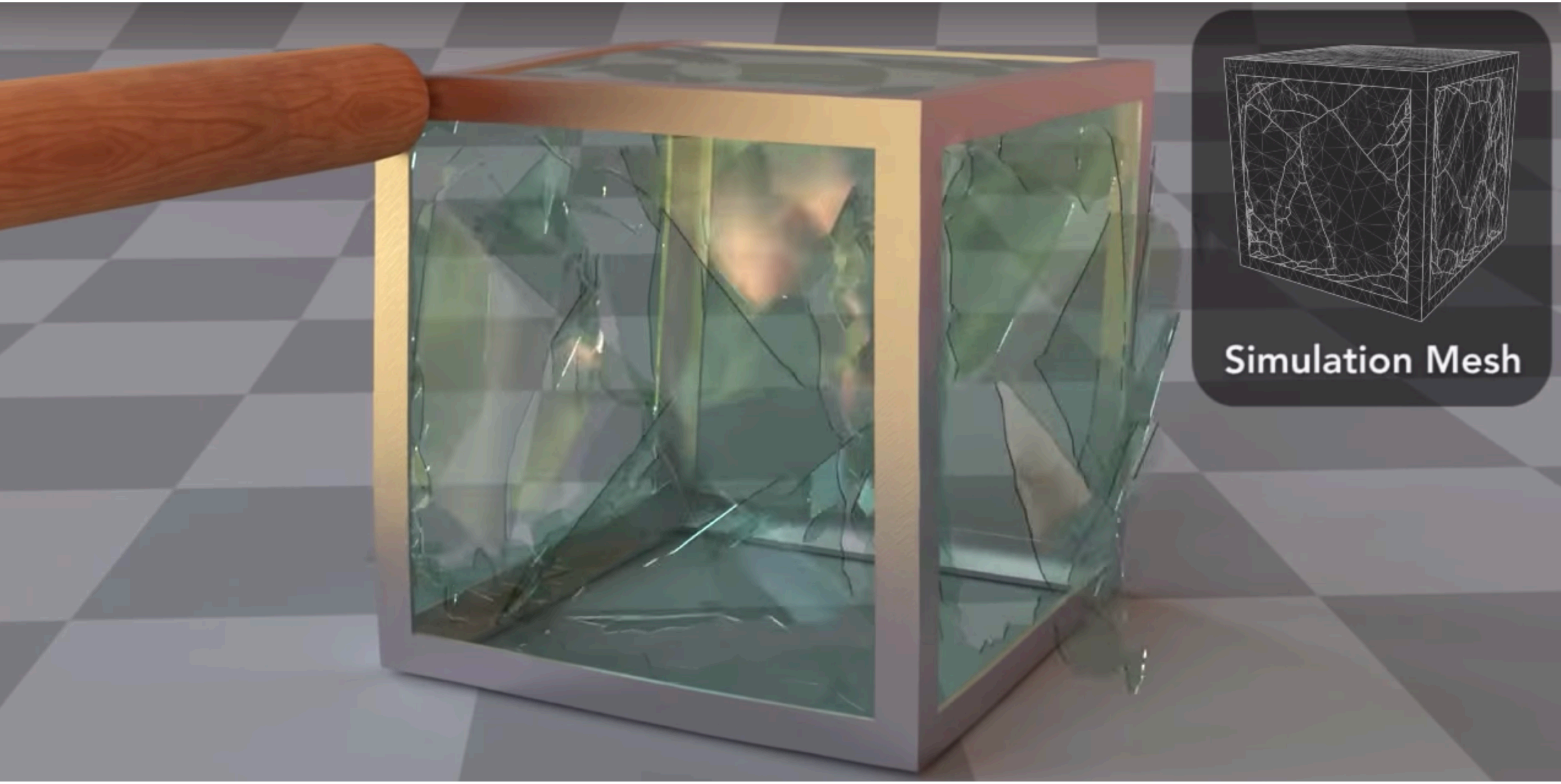
Bridson et al. 2002

Yarn-level cloth simulation



Kaldor et al. 2010

Material fracture



Pfaff et al. 2014

Summary

- Mathematical modeling of dynamical systems and (usually) solution by numerical integration
- Particle systems
  - Flexible force modeling, e.g. spring-mass sytems, gravitational attraction, fluids, flocking behavior
  - Newtonian equations of motion = ODEs
  - Solution by numerical integration of ODEs: Explicit Euler, Implicit Euler, Symplectic Euler, etc..
  - Error and instability, methods to combat instability

- *Acknowledgements: thanks to Keenan Crane, Ren Ng, Tom Funkhouser, James O'Brien for presentation resources*