Lecture 8:

Geometric Queries

Interactive Computer Graphics
Stanford CS248, Winter 2020

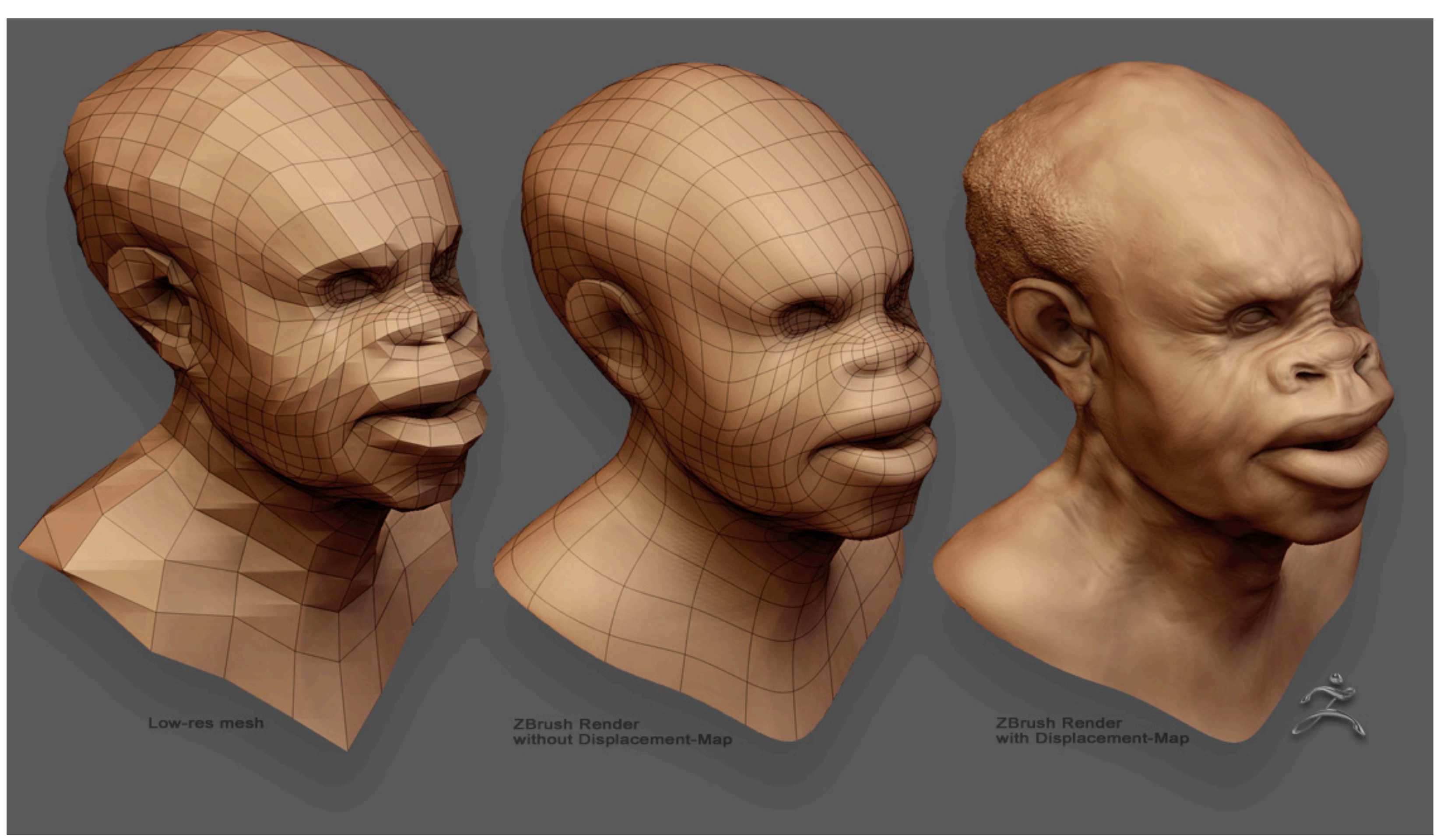
Tunes

Cake
"The Distance"
(Fashion Nugget)

"After understand the vector form of point-line and point-plane distance computations, I decided to write a song."
- John McCrea

Geometric queries — motivation

Intersecting rays and triangles
(ray tracing)

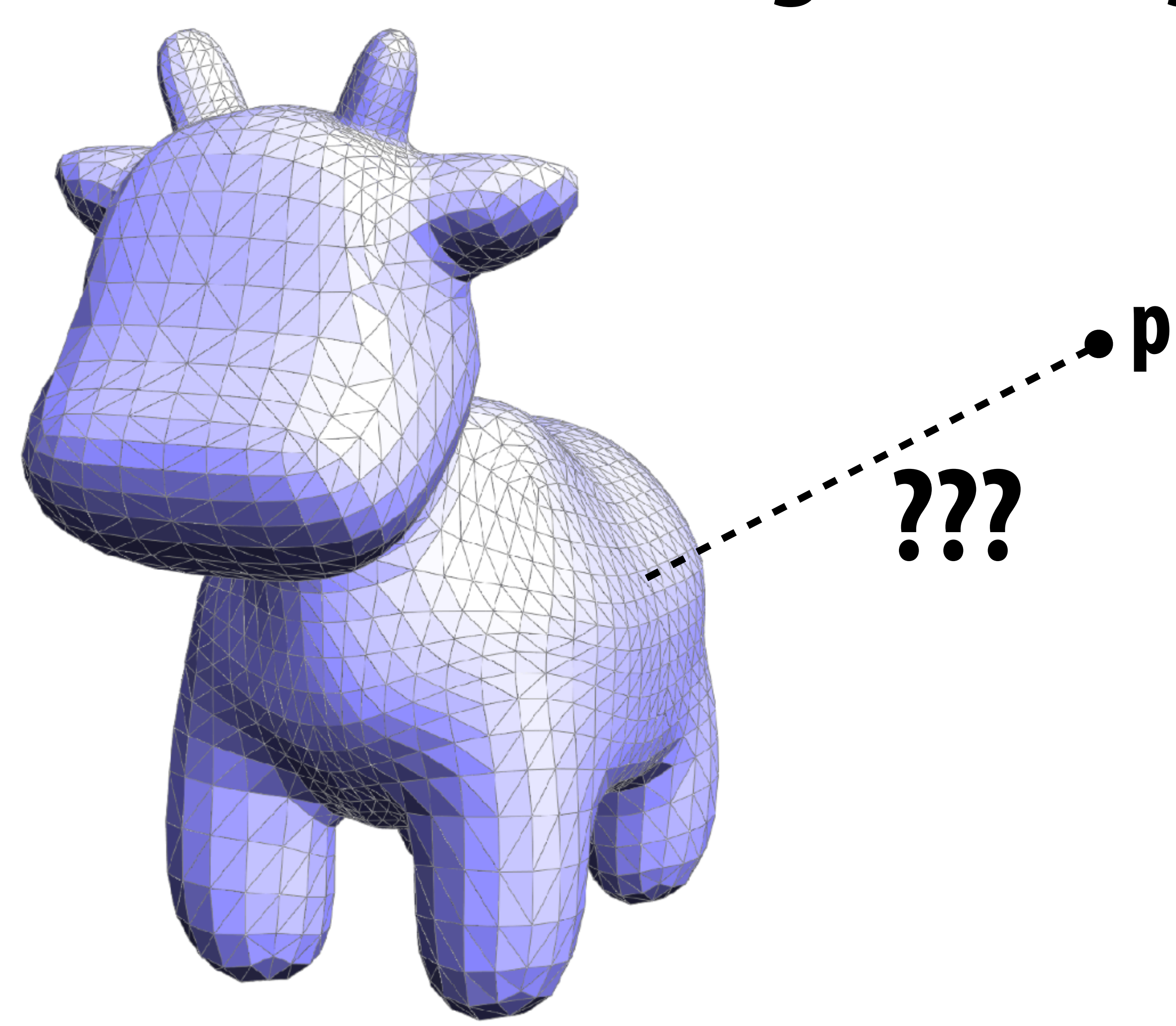


Closest point on surface queries

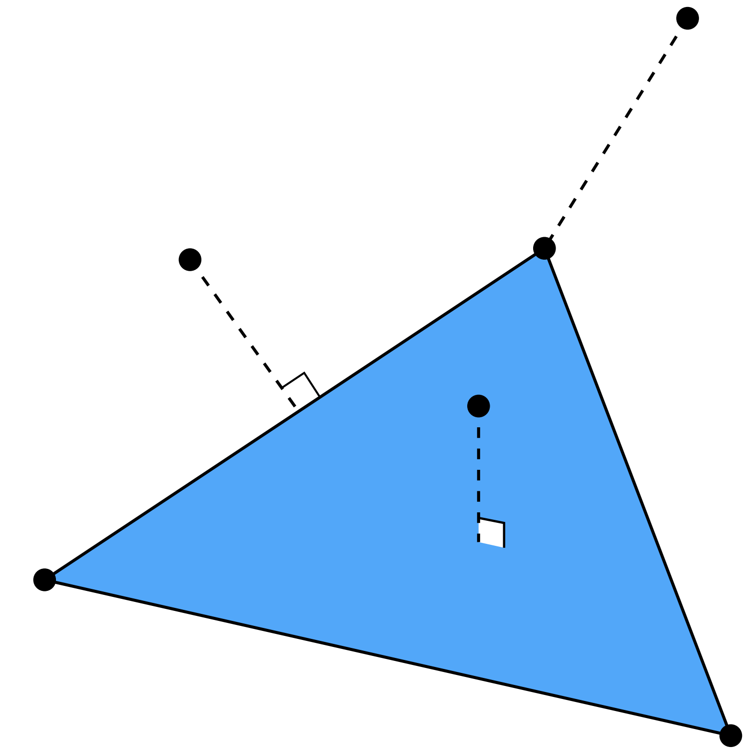
Intersecting triangles (collisions)

Example: closest point queries

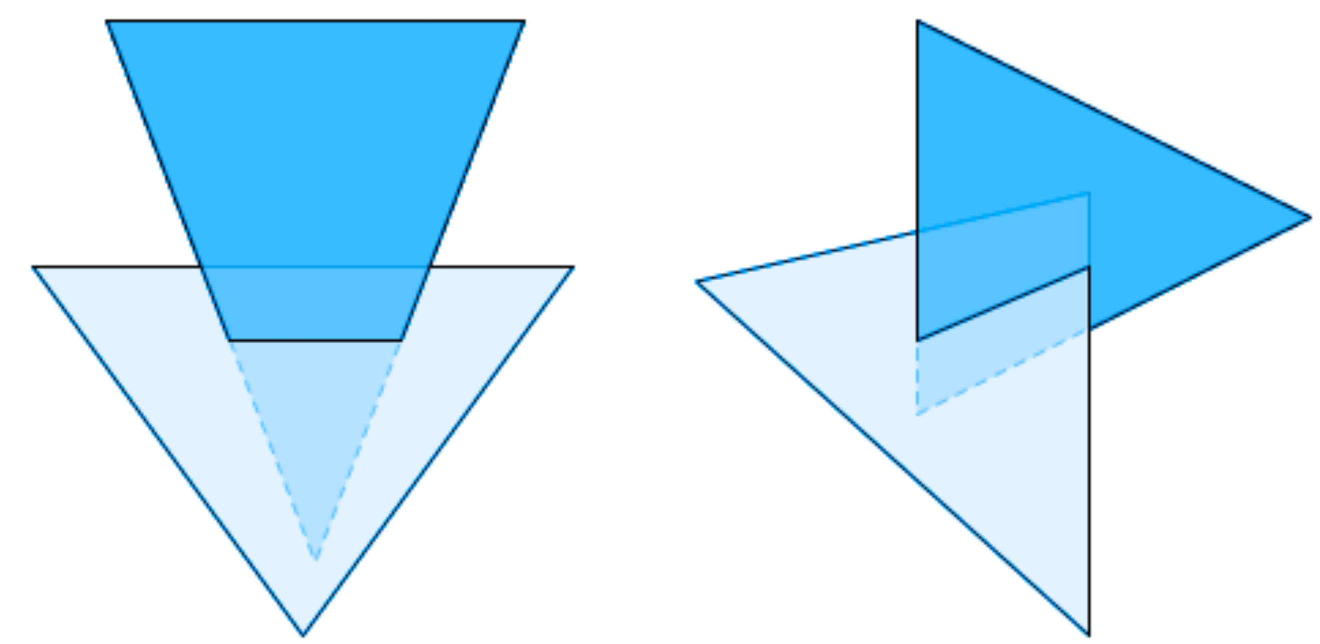
- Q: Given a point, in space (e.g., a new sample point), how do we find the closest point on a given surface?
- Q: Does implicit/explicit representation make this easier?
- Q: Does our half-edge data structure help?
- Q: What's the cost of the naïve algorithm?
- Q: How do we find the distance to a single triangle anyway?



Many types of geometric queries



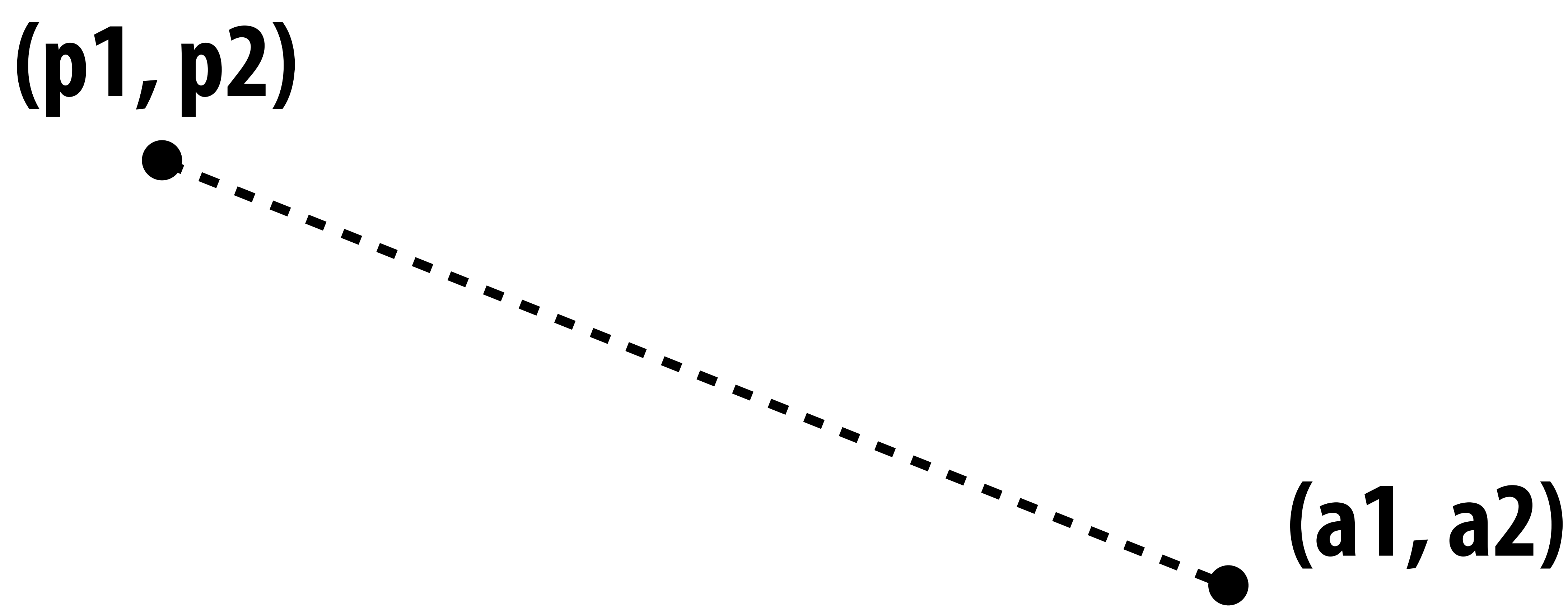
- Plenty of other things we might like to know:
  - Do two triangles intersect?
  - Are we inside or outside an object?
  - Does one object contain another?
  - ...



- Data structures we've seen so far not really designed for this...
- Need some new ideas!
- TODAY: come up with simple (aka: slow) algorithms
- NEXT TIME: intelligent ways to accelerate geometric queries

Warm up: closest point on point

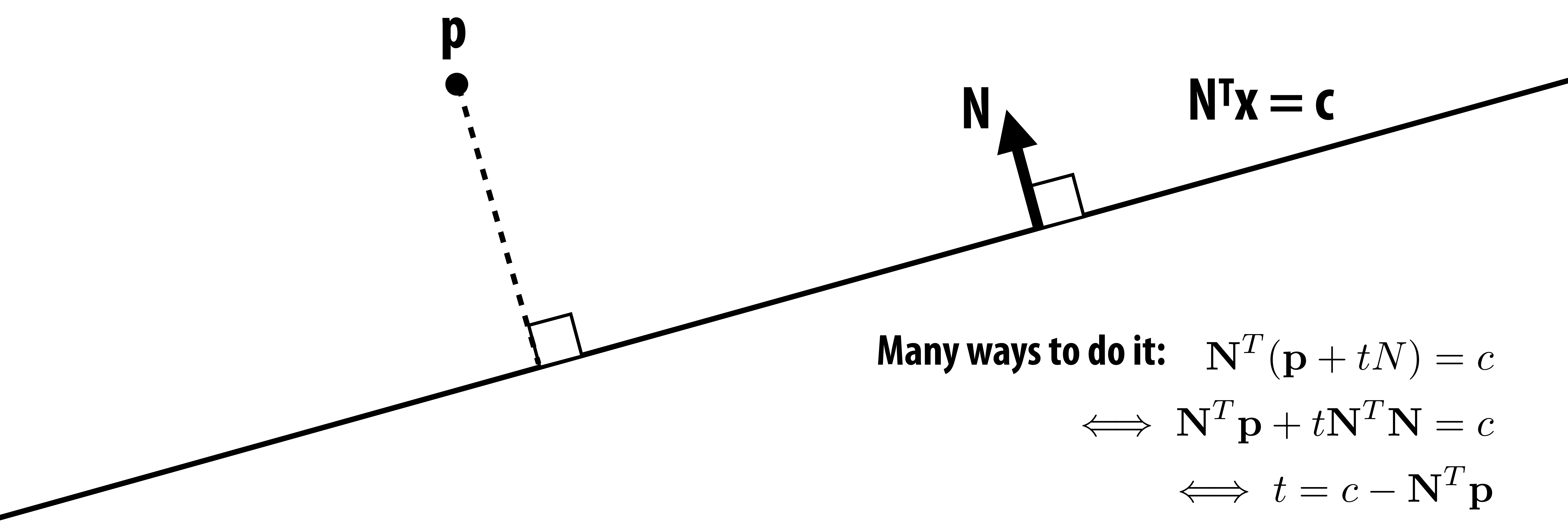
- *G*iven a query point (p1,p2), how do we find the closest point on the point (a1,a2)?



Bonus question: what's the distance?

Slighty harder: closest point on line

- Now suppose I have a line $\mathbf{N}^T\mathbf{x} = c$, where N is the unit normal
- How do I find the point on line closest to my query point p?



Many ways to do it:

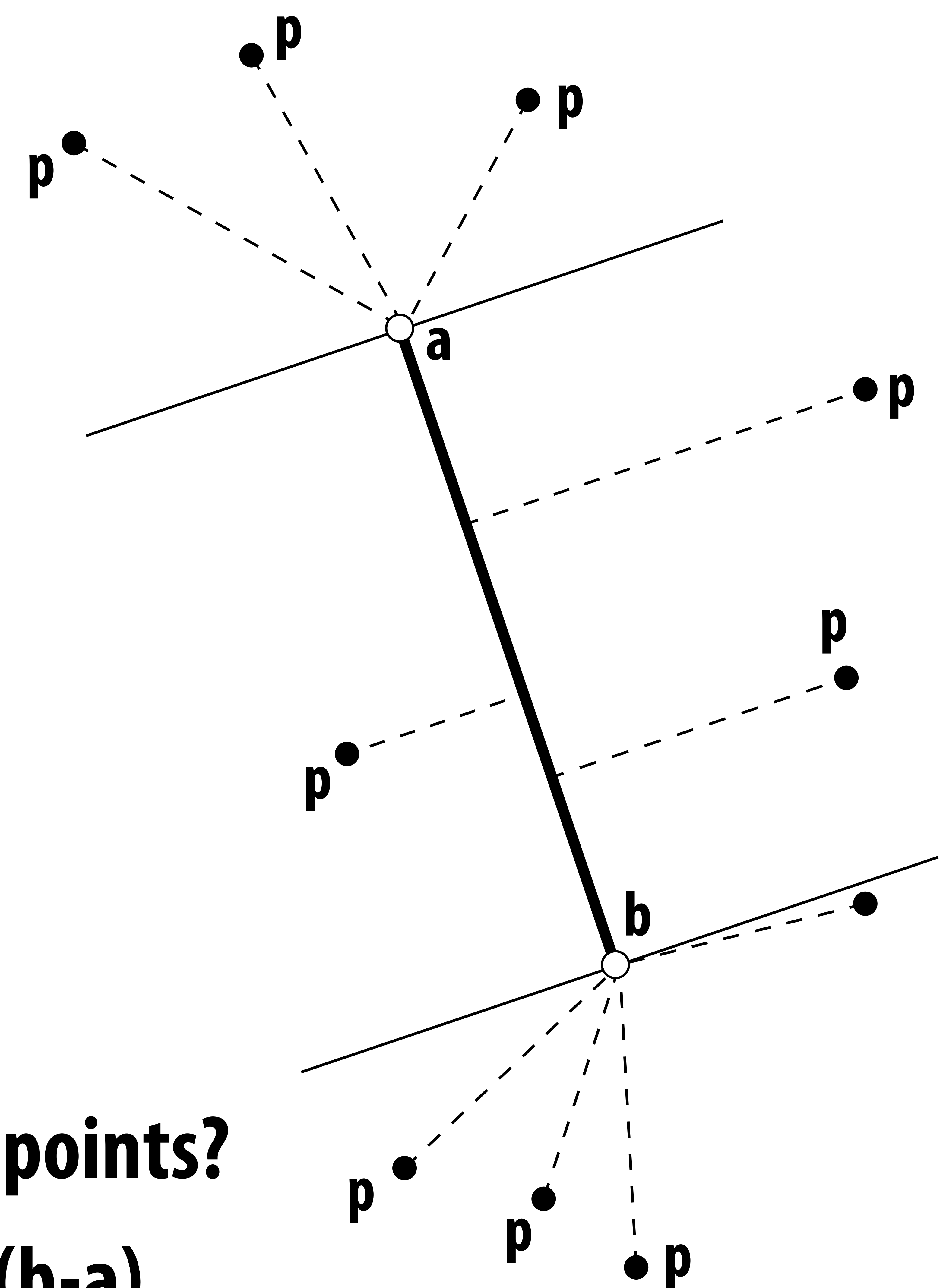
$$\mathbf{N}^T(\mathbf{p} + tN) = c$$

$$\iff \mathbf{N}^T\mathbf{p} + t\mathbf{N}^T\mathbf{N} = c$$

$$\iff t = c - \mathbf{N}^T\mathbf{p}$$

$$\Rightarrow \mathbf{p} + t\mathbf{N} = \boxed{\mathbf{p} + (c - \mathbf{N}^T\mathbf{p})\mathbf{N}}$$

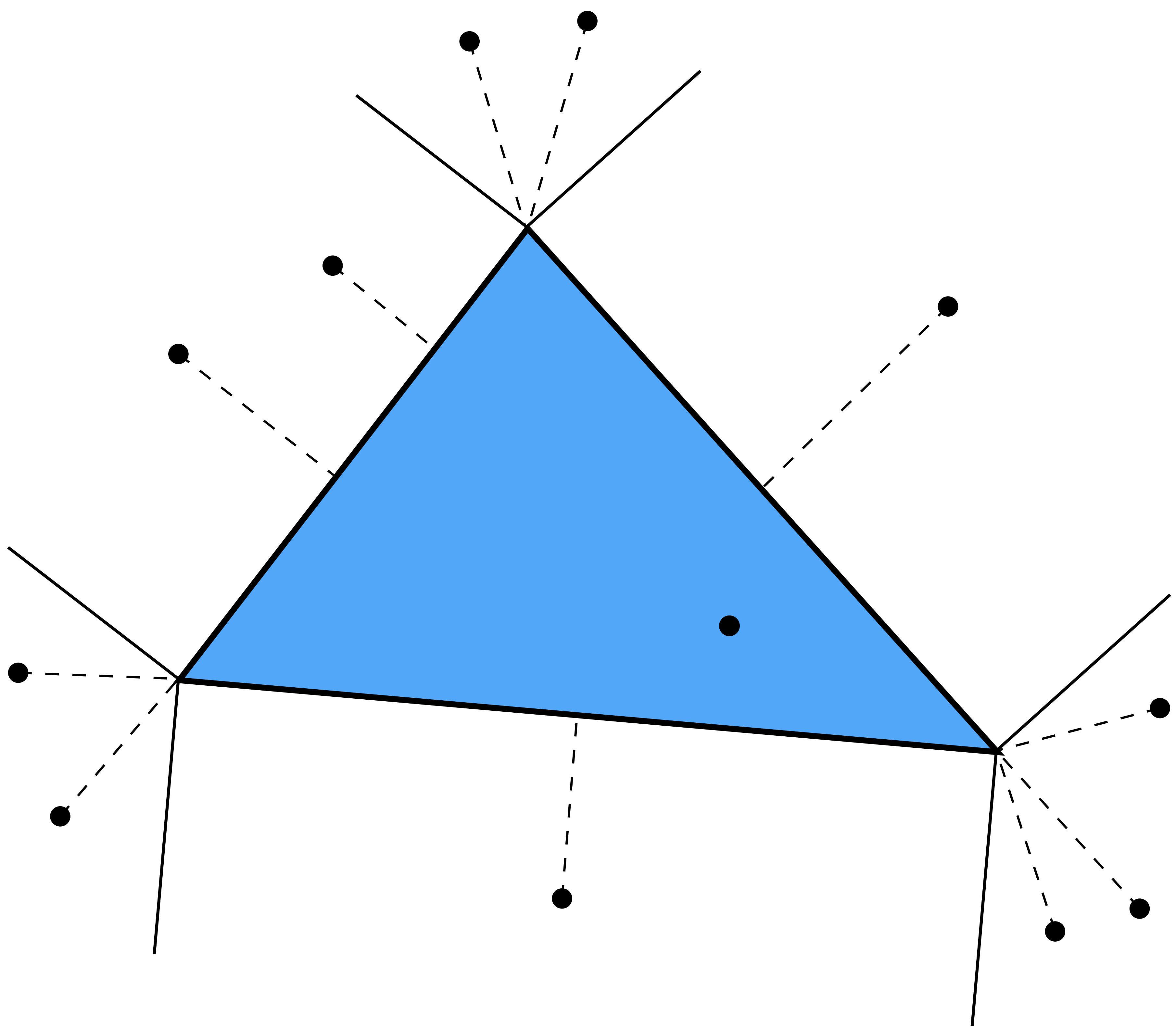
Harder: closest point on line segment



- Two cases: endpoint or interior
- Already have basic components:
  - point-to-point
  - point-to-line
- Algorithm?
  - find closest point on line
  - check if it is between endpoints
  - if not, take closest endpoint
- How do we know if it's between endpoints?
  - write closest point on line as a+t(b-a)
  - if t is between 0 and 1, it's inside the segment!

Even harder: closest point on triangle in 2D

- What are all the possibilities for the closest point?
- Almost just minimum distance to three line segments:



Q: What about a point inside the triangle?

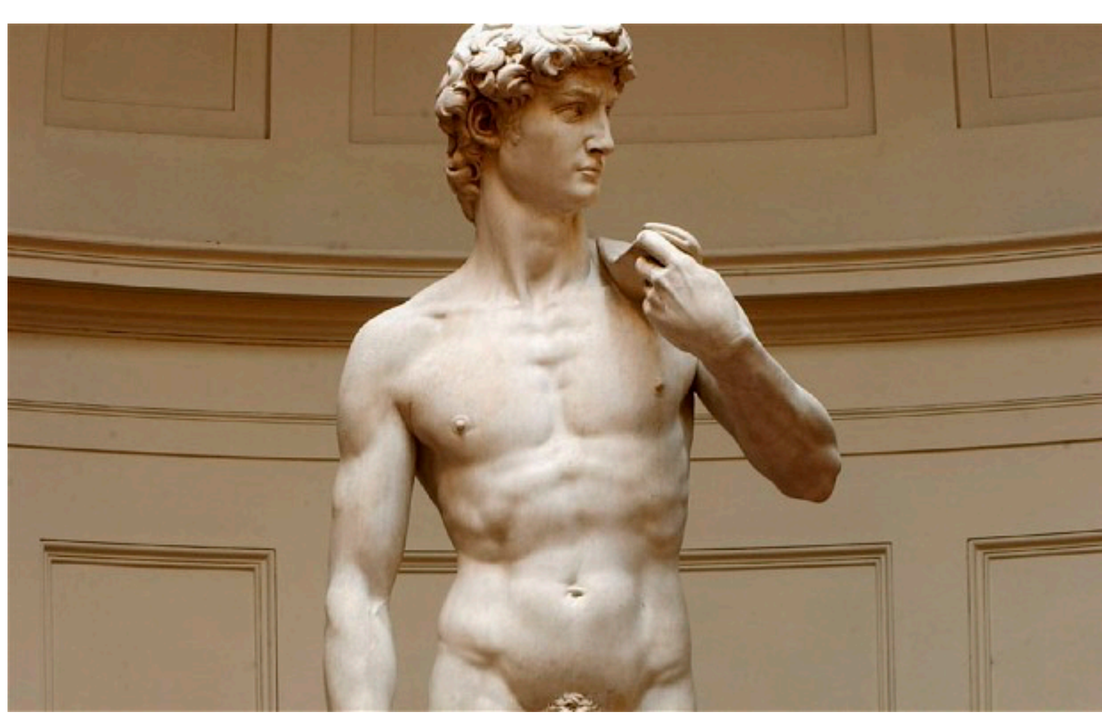
Closest point on triangle in 3D

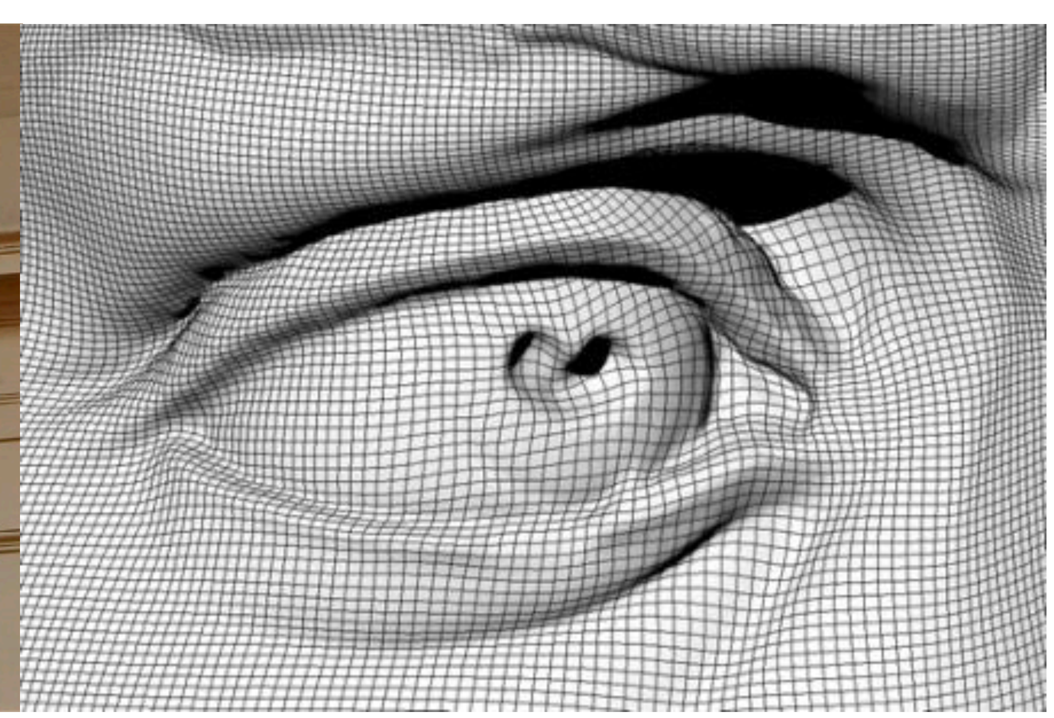
- Not so different from 2D case
- Algorithm:
  - Project point onto plane of triangle
  - Use half-*space* tests to classify point (vs. half plane)
  - If inside the triangle, we're done!
  - Otherwise, find closest point on associated vertex or edge

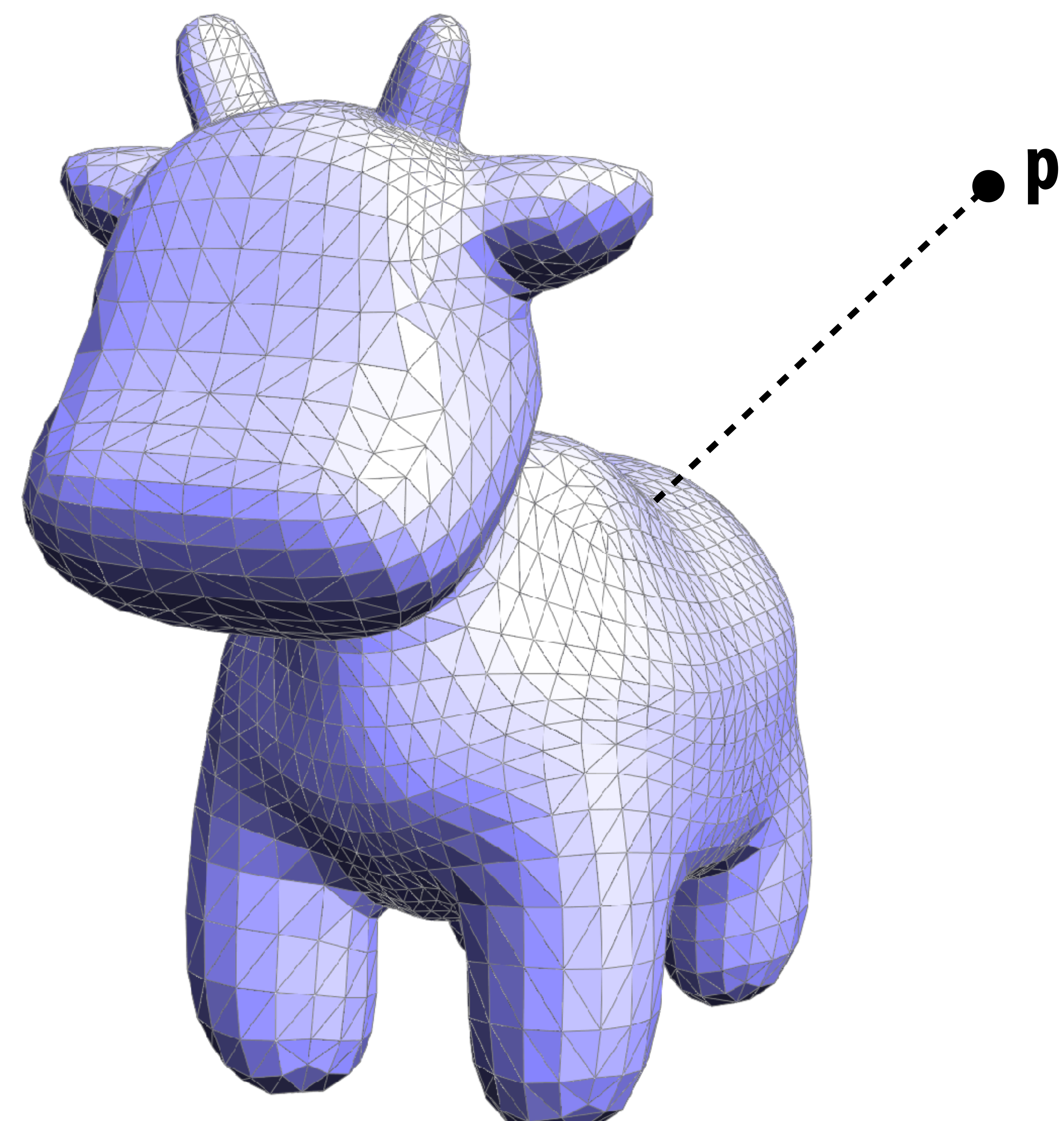
- By the way, how do we find closest point on plane?
- Same expression as closest point on a line! $p + (c - N^T p) N$

Closest point on triangle *mesh* in 3D?

- Conceptually easy:
  - loop over all triangles
  - compute closest point to current triangle
  - keep globally closest point
- Q: What's the cost?
- What if we have *billions* of faces?
- NEXT TIME: Better data structures!

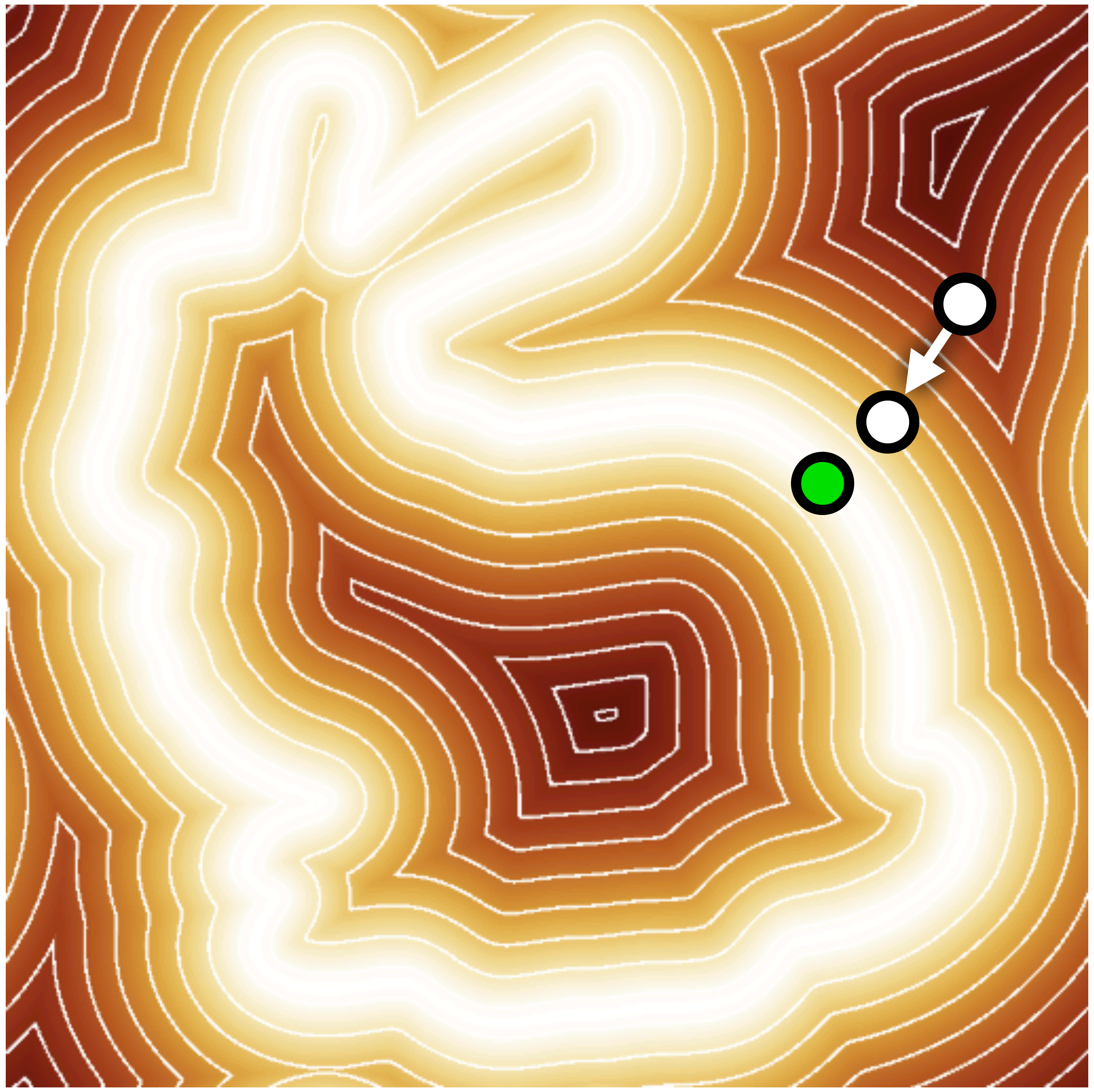






Closest point to *implicit* surface?

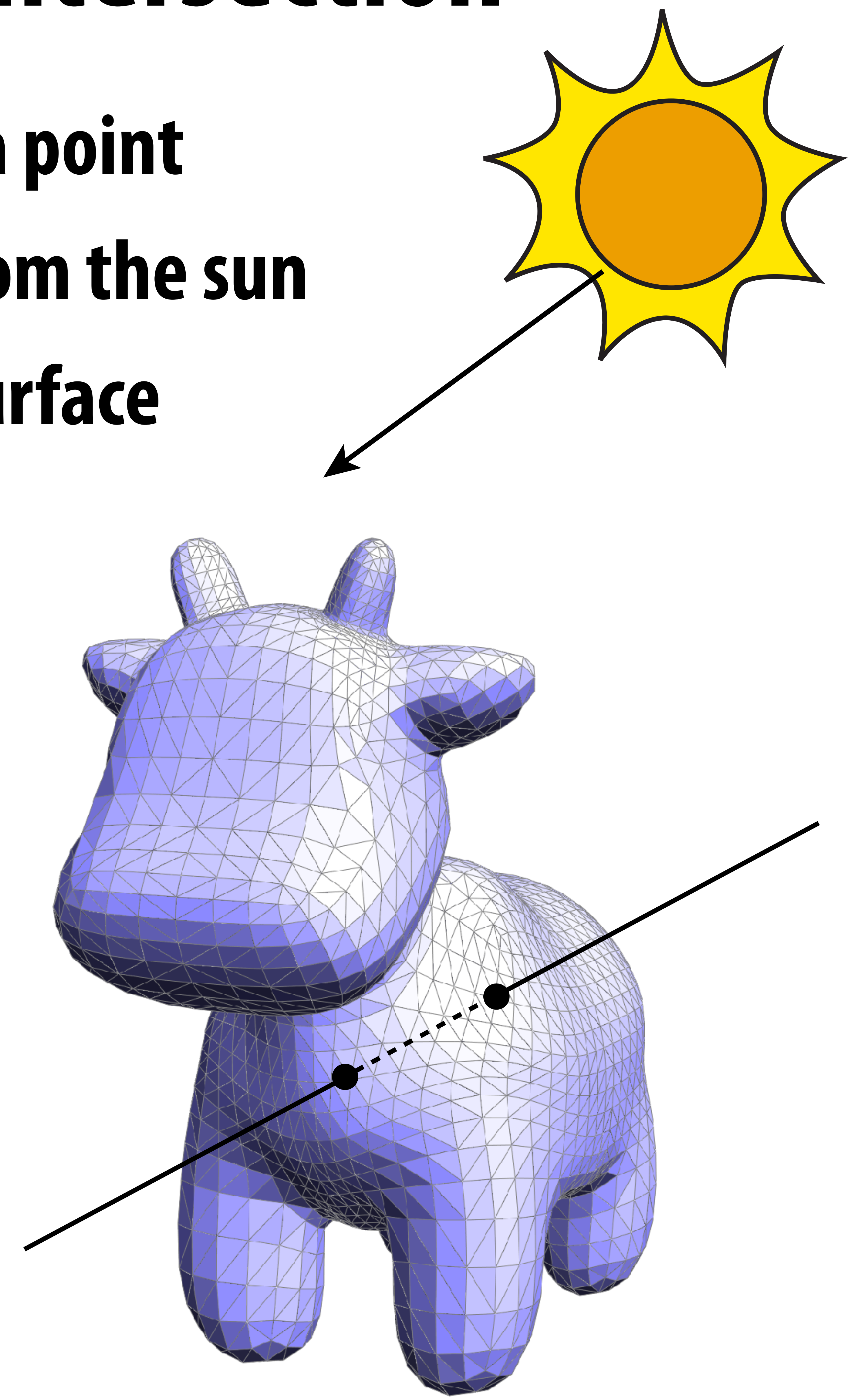
- If we change our representation of geometry, algorithms can change completely
- E.g., how might we compute the closest point on an implicit surface described via its distance function?
- One idea:
  - start at the query point
  - compute gradient of distance (using, e.g., finite differences)
  - take a little step (decrease distance)
  - repeat until we're at the surface (zero distance)



- Better yet: just store closest point for each grid cell! (speed/memory trade off)

Different query: ray-mesh intersection

- A "ray" is an oriented line starting at a point
- Think about a ray of light traveling from the sun
- Want to know where a ray pierces a surface
- Why?
  - GEOMETRY: inside-outside test
  - RENDERING: visibility, ray tracing
  - ANIMATION: collision detection
- Might pierce surface in many places!



Ray equation

- **Can express ray as**

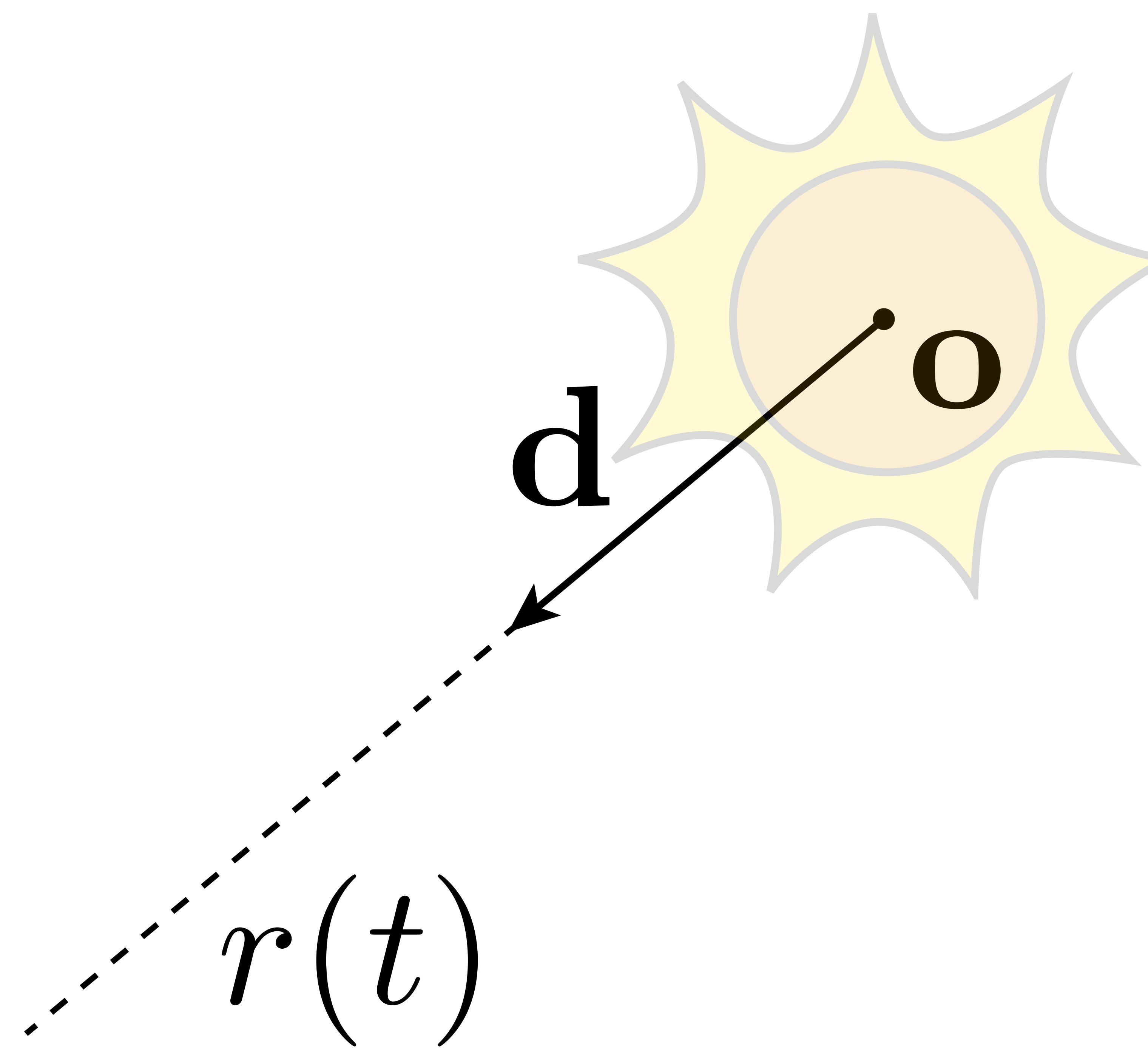
origin

unit direction

$$\mathbf{r}(t) = \mathbf{o} + t\mathbf{d}$$

point along ray

"time"



Intersecting a ray with an implicit surface

- Recall implicit surfaces: all points x such that f(x) = 0
- Q: How do we find points where a ray pierces this surface?
- Well, we know all points along the ray: r(t) = o + td
- Idea: replace "x" with "r" in 1st equation, and solve for t
- Example: unit sphere

$$f(\mathbf{x}) = |\mathbf{x}|^2 - 1$$

$$\Rightarrow f(\mathbf{r}(t)) = |\mathbf{o} + t\mathbf{d}|^2 - 1$$

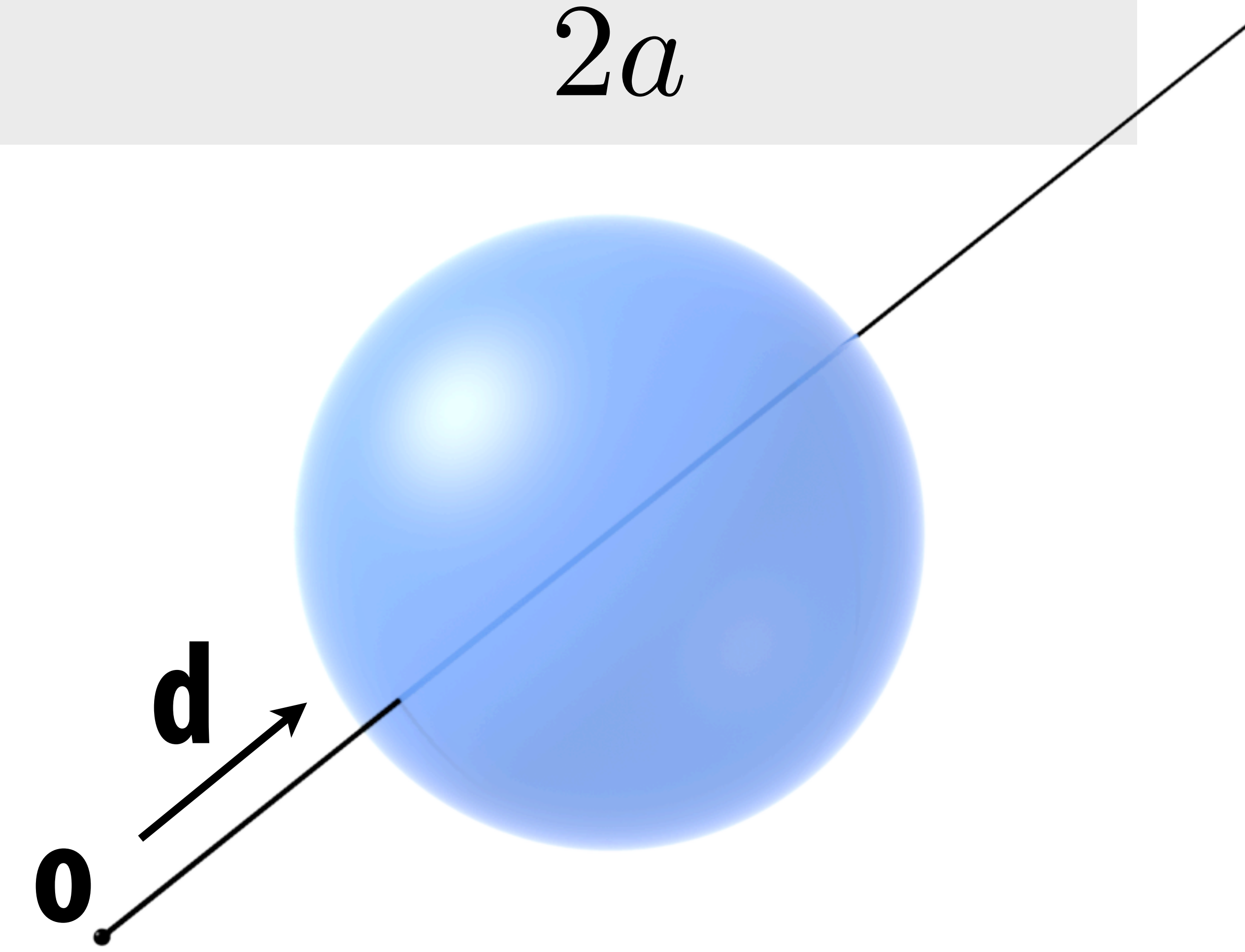
$$\underbrace{|\mathbf{d}|^2}_{a} t^2 + \underbrace{2(\mathbf{o}\cdot\mathbf{d})}_{b} t + \underbrace{|\mathbf{o}|^2 - 1}_{c} = 0$$

Note: $|\mathrm{d}|^2 = 1$ since d is a unit vector

$$t = \boxed{-\mathbf{o}\cdot\mathbf{d} \pm \sqrt{(\mathbf{o}\cdot\mathbf{d})^2 - |\mathbf{o}|^2 + 1}}$$

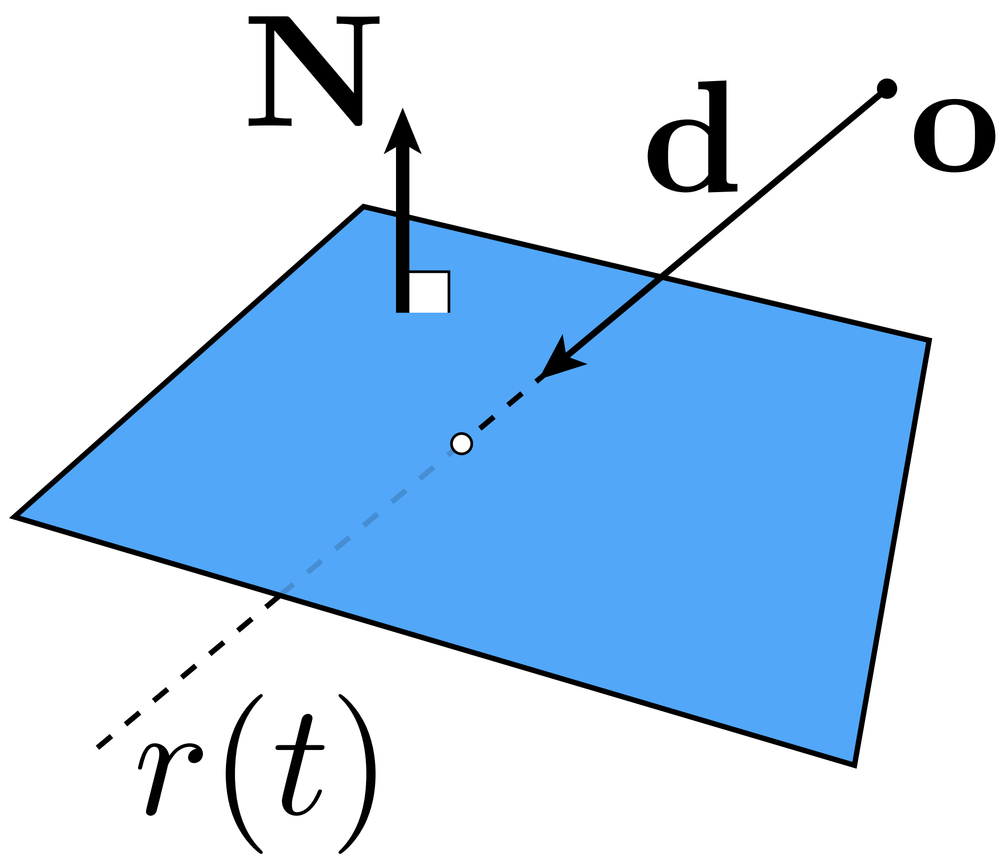
quadratic formula:

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$



Why two solutions?

Ray-plane intersection



- Suppose we have a plane $N^T x = c$
  - N - unit normal
  - c - offset
- How do we find intersection with ray $r(t) = o + td$?
- *Key idea:* again, replace the point x with the ray equation t:

$$\mathbf{N}^{\mathsf{T}}\mathbf{r}(t) = c$$

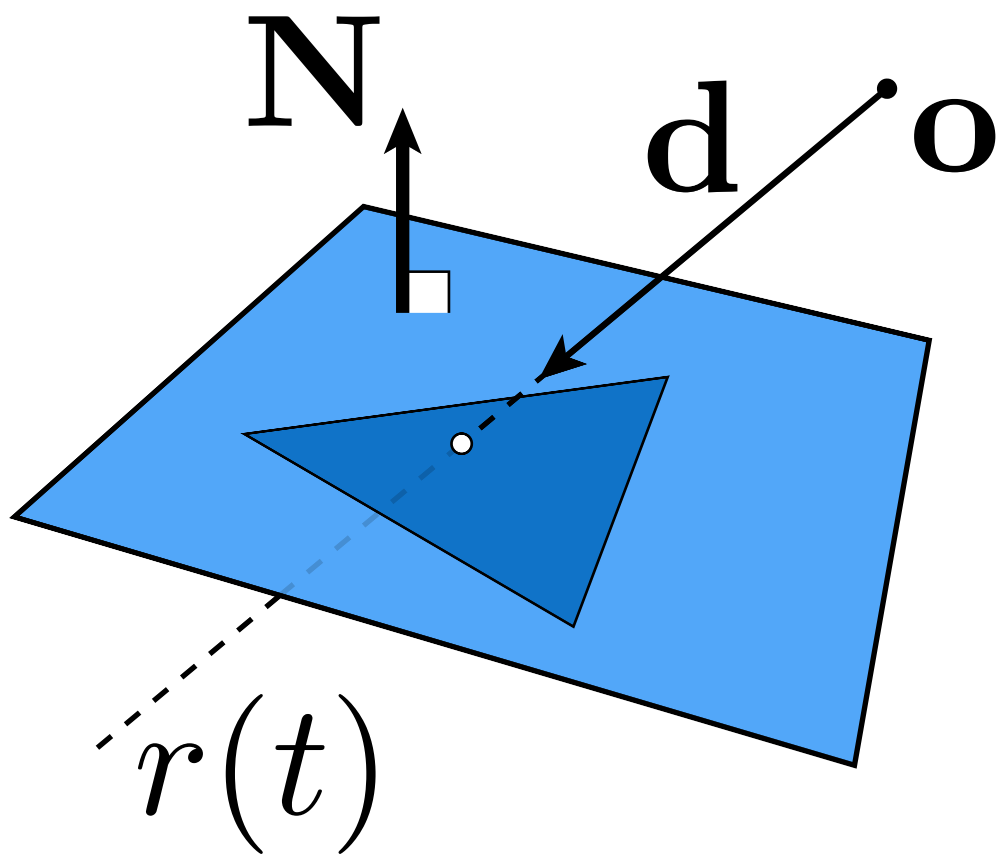
- Now solve for t:

$$\mathbf{N}^{\mathsf{T}}(\mathbf{o} + t\mathbf{d}) = c \qquad \Rightarrow t = \frac{c - \mathbf{N}^{\mathsf{T}}\mathbf{o}}{\mathbf{N}^{\mathsf{T}}\mathbf{d}}$$

- And plug t back into ray equation:

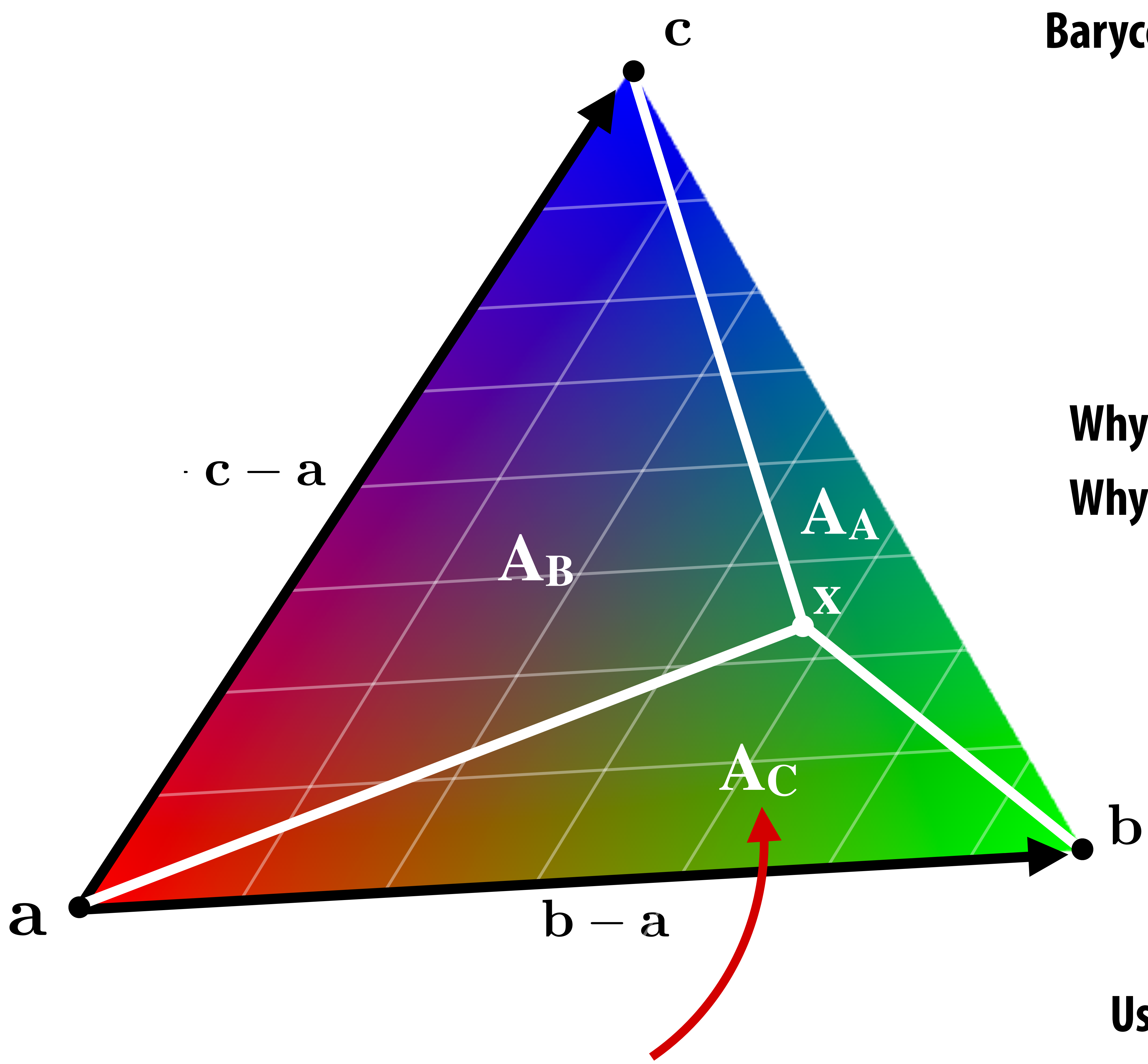
$$r(t) = \mathbf{o} + \frac{c - \mathbf{N}^{\mathsf{T}}\mathbf{o}}{\mathbf{N}^{\mathsf{T}}\mathbf{d}}\mathbf{d}$$

Ray-triangle intersection



- Triangle is in a plane...
- Algorithm:
  - Compute ray-plane intersection
  - Q: What do we do now?

Barycentric coordinates (as ratio of areas)



Area of triangle formed by points: a, b, x

Barycentric coords are *signed* areas:

$$\alpha = A_A/A$$

$$\beta = A_B/A$$

$$\gamma = A_C/A$$

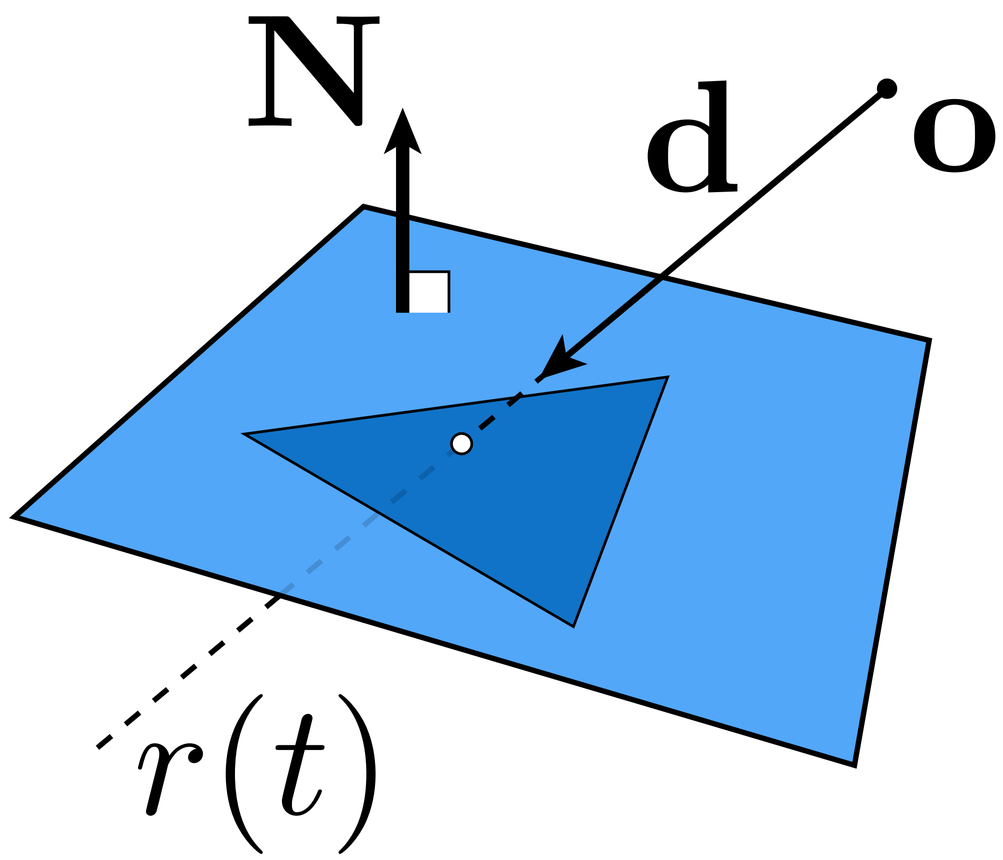
Why must coordinates sum to one?

Why must coordinates be between 0 and 1?

Useful: Heron's formula:

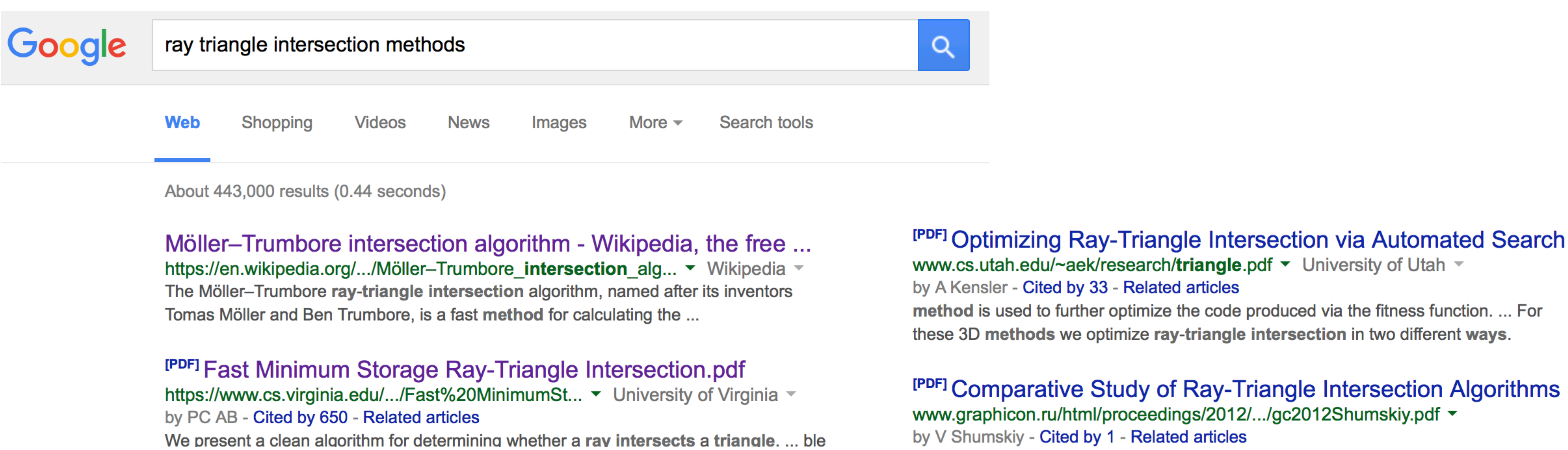
$$A_C = \frac{1}{2}(\mathbf{b}-\mathbf{a}) \times (\mathbf{x}-\mathbf{a})$$

Ray-triangle intersection



- **Algorithm:**
  - Compute ray-plane intersection
  - Q: What do we do now?
  - A: Compute barycentric coordinates of hit point?
  - If barycentric coordinates are all positive, point is in triangle

- **Many different techniques if you care about efficiency**



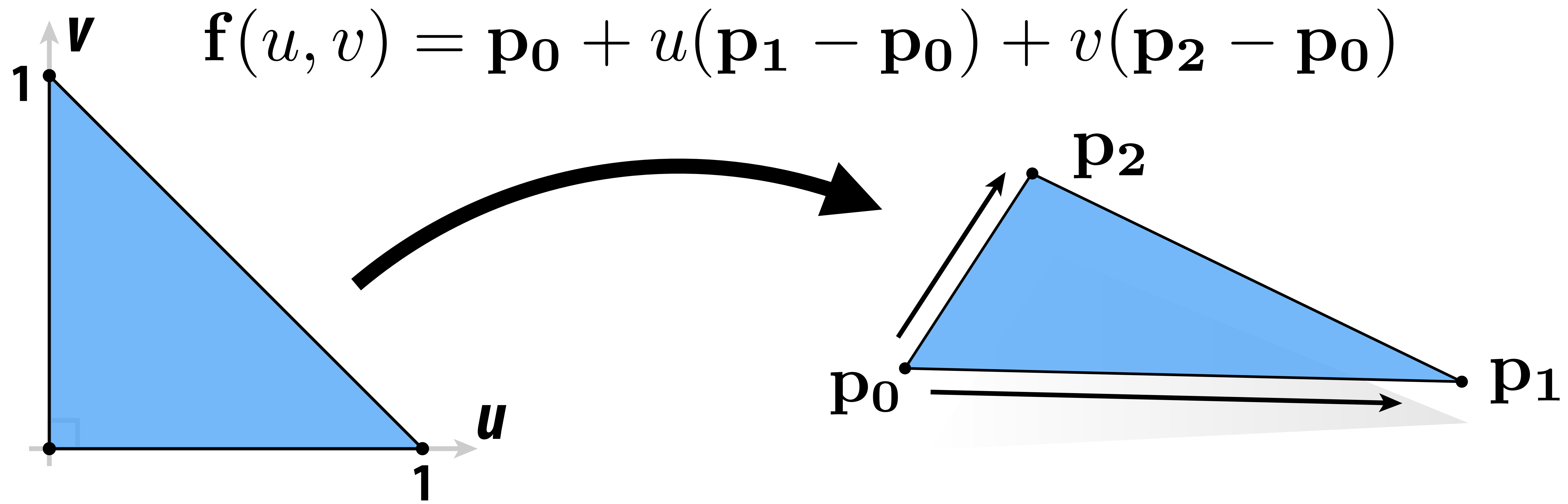
Another way: ray-triangle intersection

- **Parameterize triangle given by vertices $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2$ using barycentric coordinates**

$$f(u,v) = (1-u-v)\mathbf{p_0} + u\mathbf{p_1} + v\mathbf{p_2}$$

- **Can think of a triangle as an affine map of the unit triangle**

$$\mathbf{f}(u,v) = \mathbf{p_0} + u(\mathbf{p_1} - \mathbf{p_0}) + v(\mathbf{p_2} - \mathbf{p_0})$$



Another way: ray-triangle intersection

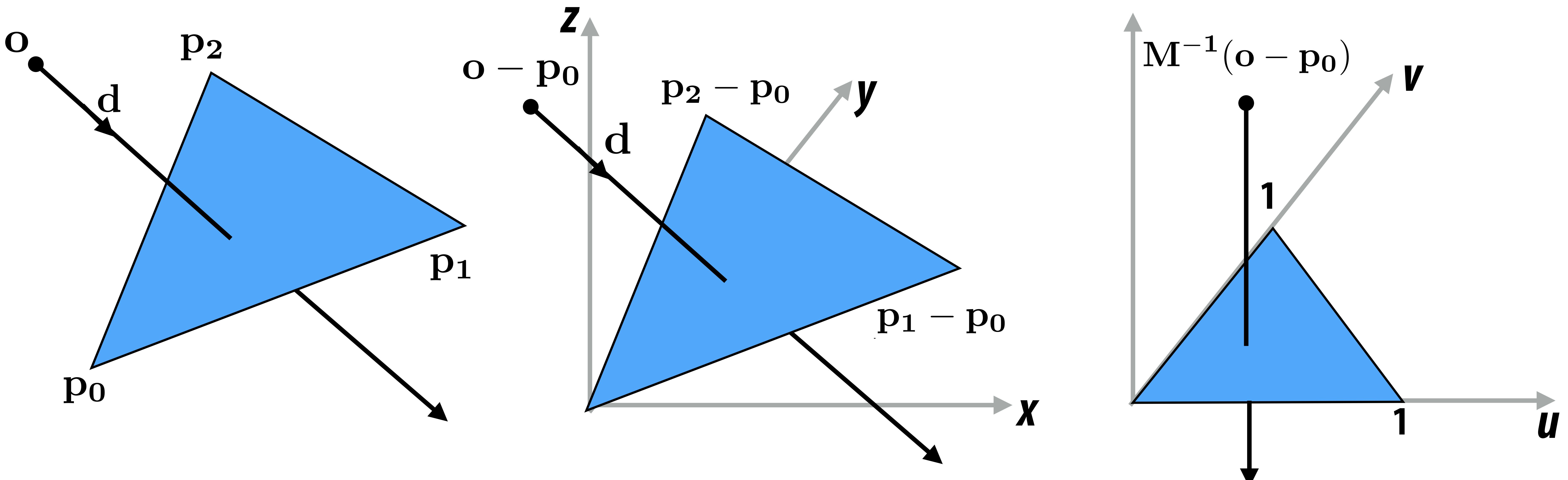
Plug parametric ray equation directly into equation for points on triangle:

$$\mathbf{p_0} + u(\mathbf{p_1} - \mathbf{p_0}) + v(\mathbf{p_2} - \mathbf{p_0}) = \mathbf{o} + t\mathbf{d}$$

Solve for u, v, t:

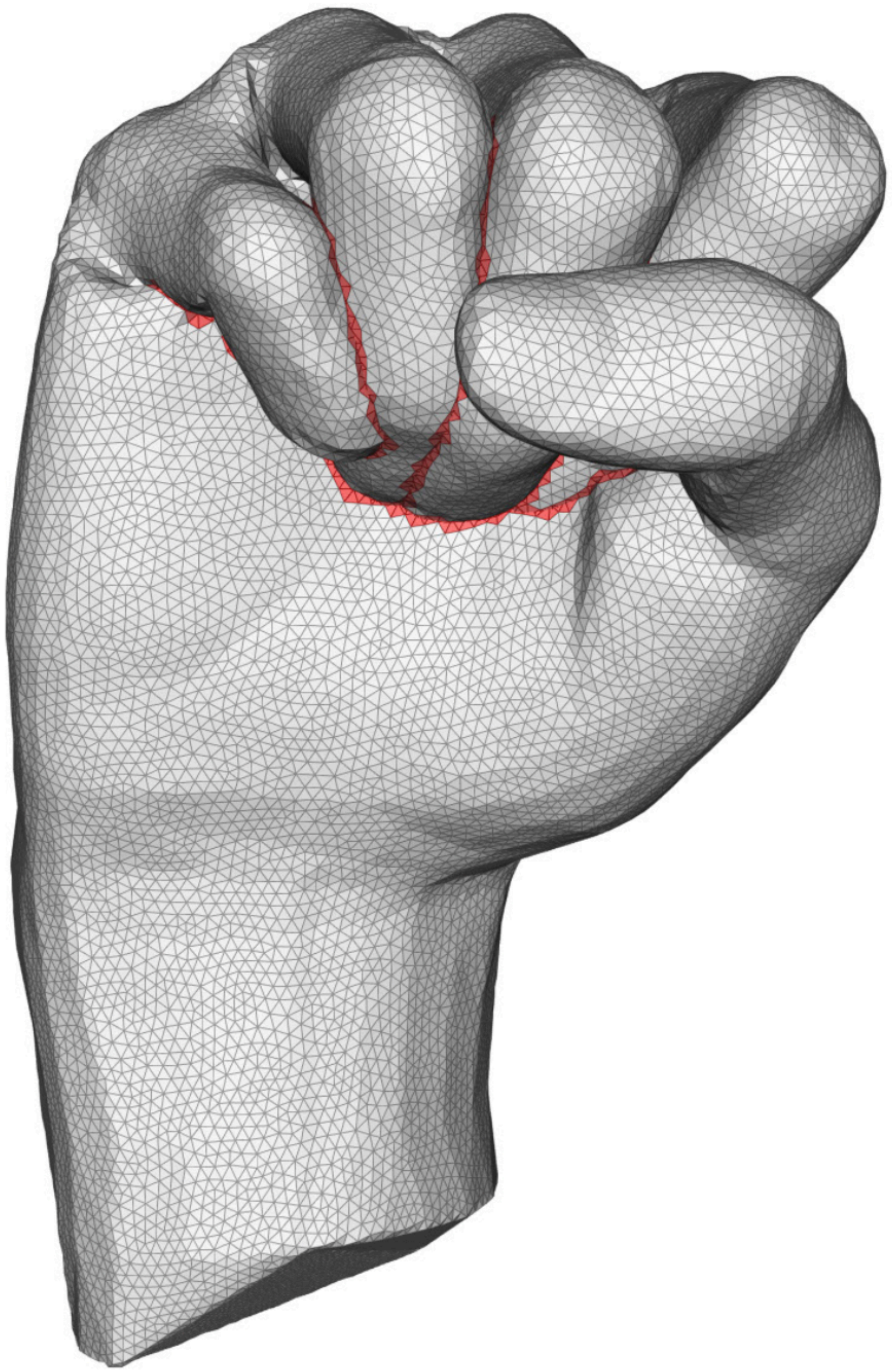
$$\underbrace{\begin{bmatrix}\mathbf{p_1} - \mathbf{p_0} & \mathbf{p_2} - \mathbf{p_0} & -\mathbf{d}\end{bmatrix}}_{\mathbf{M}} \begin{bmatrix} u \\ v \\ t \end{bmatrix} = \mathbf{o} - \mathbf{p_0}$$

$\mathbf{M}^{-1}$ **transforms triangle back to unit triangle in u,v plane, and transforms ray's direction to be orthogonal to plane. It's a point in 2D triangle test now!**



One more query: mesh-mesh intersection

- **GEOMETRY: How do we know if a mesh intersects itself?**
- **ANIMATION: How do we know if a collision occurred?**

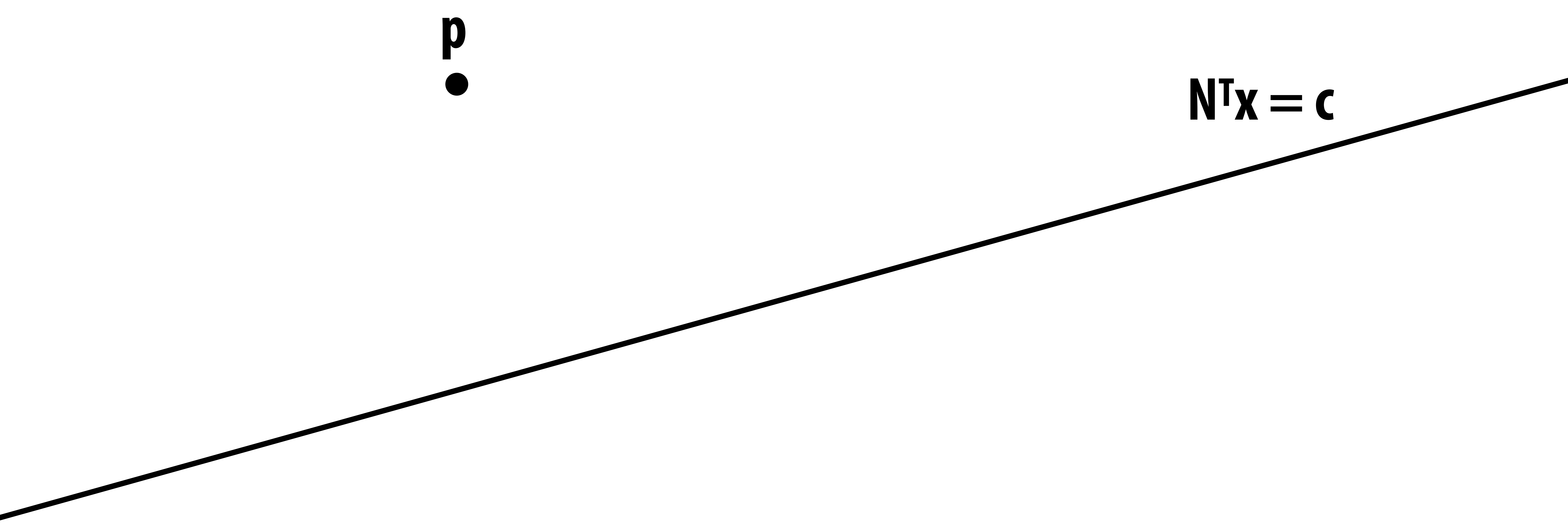


Warm up: point-point intersection

- Q: How do we know if p intersects a?
- A: …check if they're the same point!

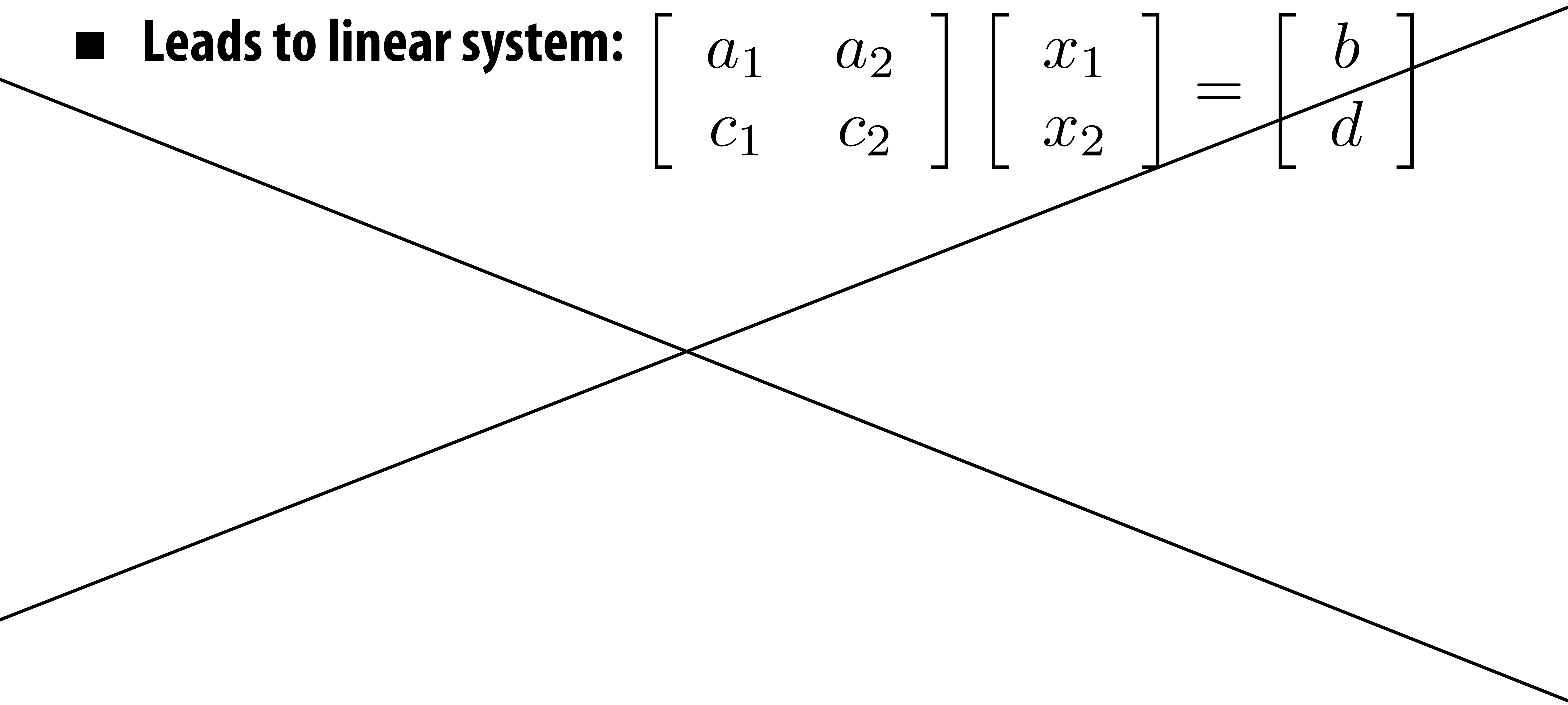
Slightly harder: point-line intersection

- Q: How do we know if a point intersects a given line?
- A: ...plug it into the line equation!



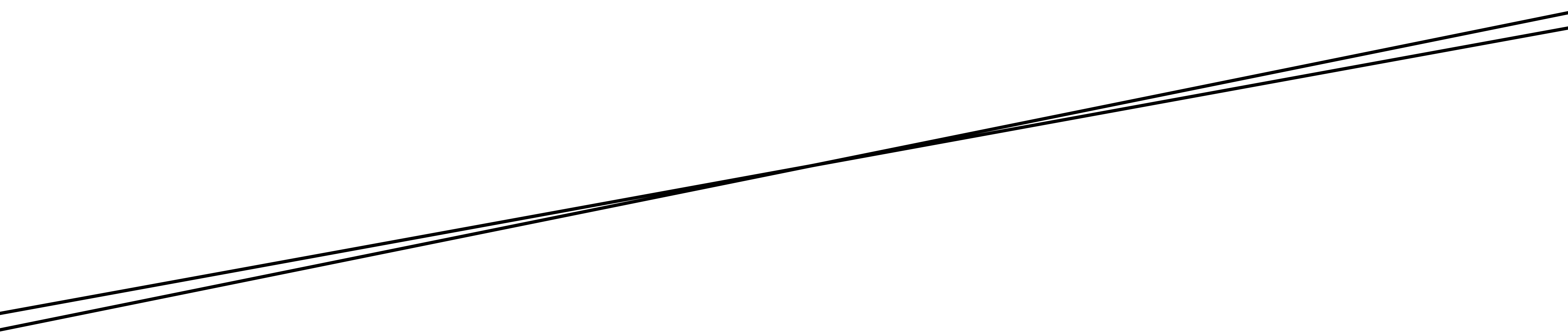
Line-line intersection

- **Two lines: ax=b and cx=d**
- **Q: How do we find the intersection?**
- **A: See if there is a simultaneous solution**
- **Leads to linear system:** $$\begin{bmatrix} a_1 & a_2 \\ c_1 & c_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$$



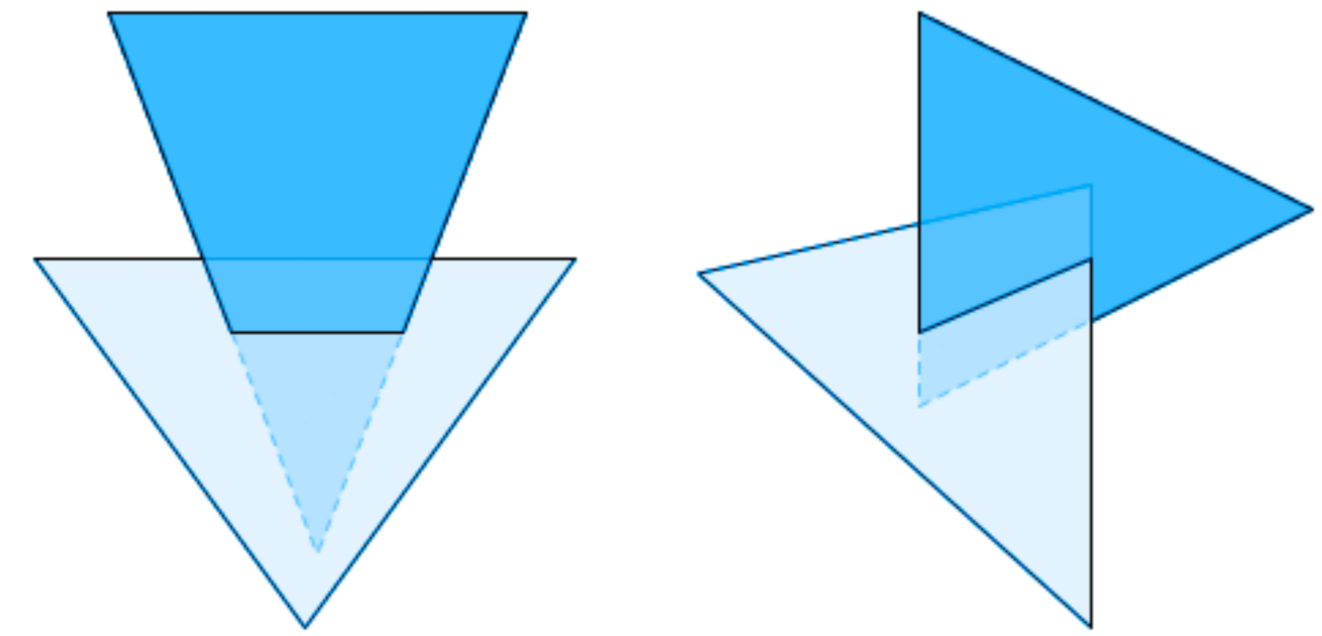
Degenerate line-line intersection?

- **What if lines are almost parallel?**
- **Small change in normal can lead to big change in intersection!**
- **Instability very common, very important with geometric predicates. Demands special care (e.g., analysis of matrix).**

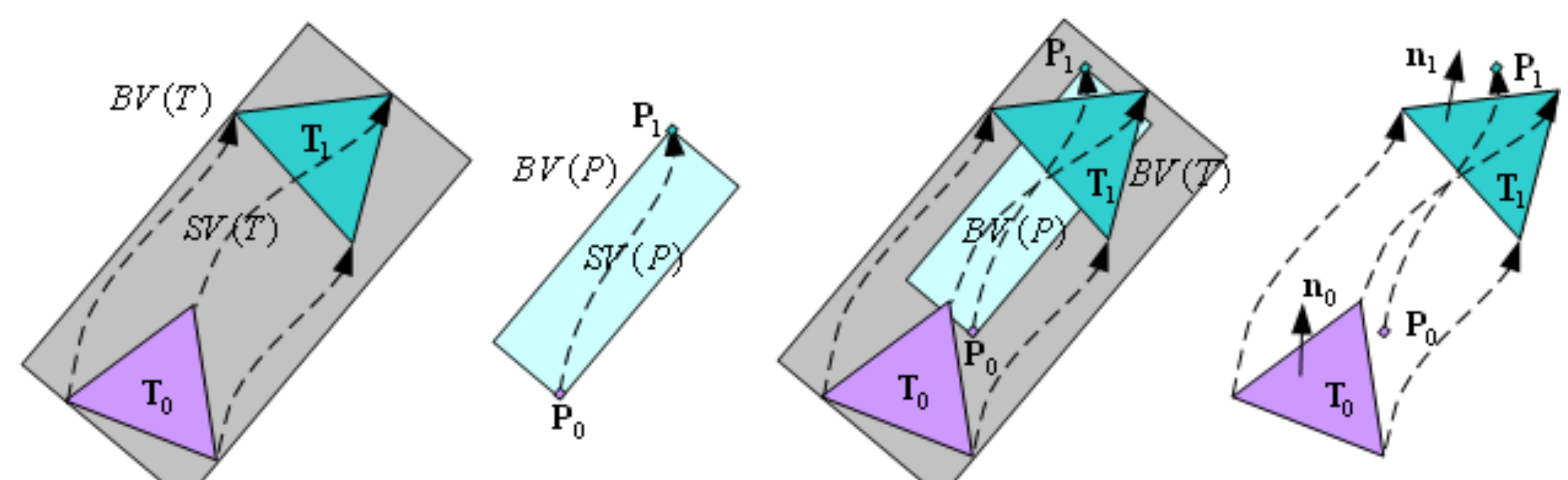


See for example Shewchuk, "*Adaptive Precision Floating-Point Arithmetic and Fast Robust Geometric Predicates*"

Triangle-triangle intersection?



- Lots of ways to do it
- Basic idea:
  - Q: Any ideas?
  - One way: reduce to edge-triangle intersection
  - Check if each line passes through plane (ray-triangle)
  - Then do interval test



(a) Bounding volume of a deforming triangle
(b) Bounding volume of a deforming vertex
(c) Bounding volume test
(d) Coplanarity test

- What if triangle is *moving*?
  - Important case for animation
  - Can think of triangles as *prisms* in time
  - Turns dynamic problem (in nD + time) into purely geometric problem in (n+1)-dimensions

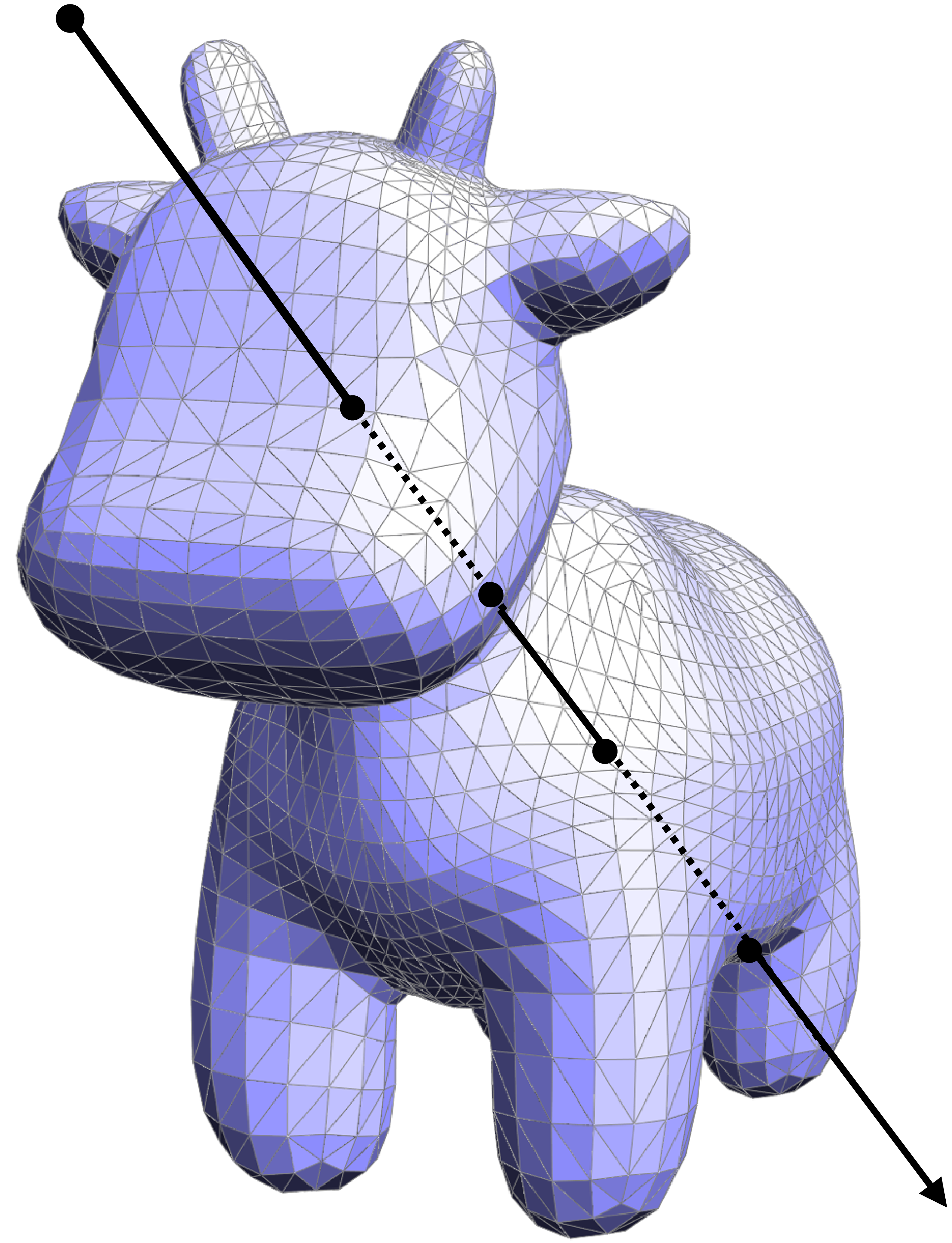
Ray-scene intersection

Given a scene defined by a set of *N* primitives and a ray *r*, find the closest point of intersection of *r* with the scene

"Find the first primitive the ray hits"

```
p_closest = NULL
t_closest = inf
for each primitive p in scene:
   t = p.intersect(r)
   if t >= 0 && t < t_closest:
      t_closest = t
      p_closest = p
```

(Assume p.intersect(r) returns value of *t* corresponding to the point of intersection with ray *r*)



Complexity? $O(N)$

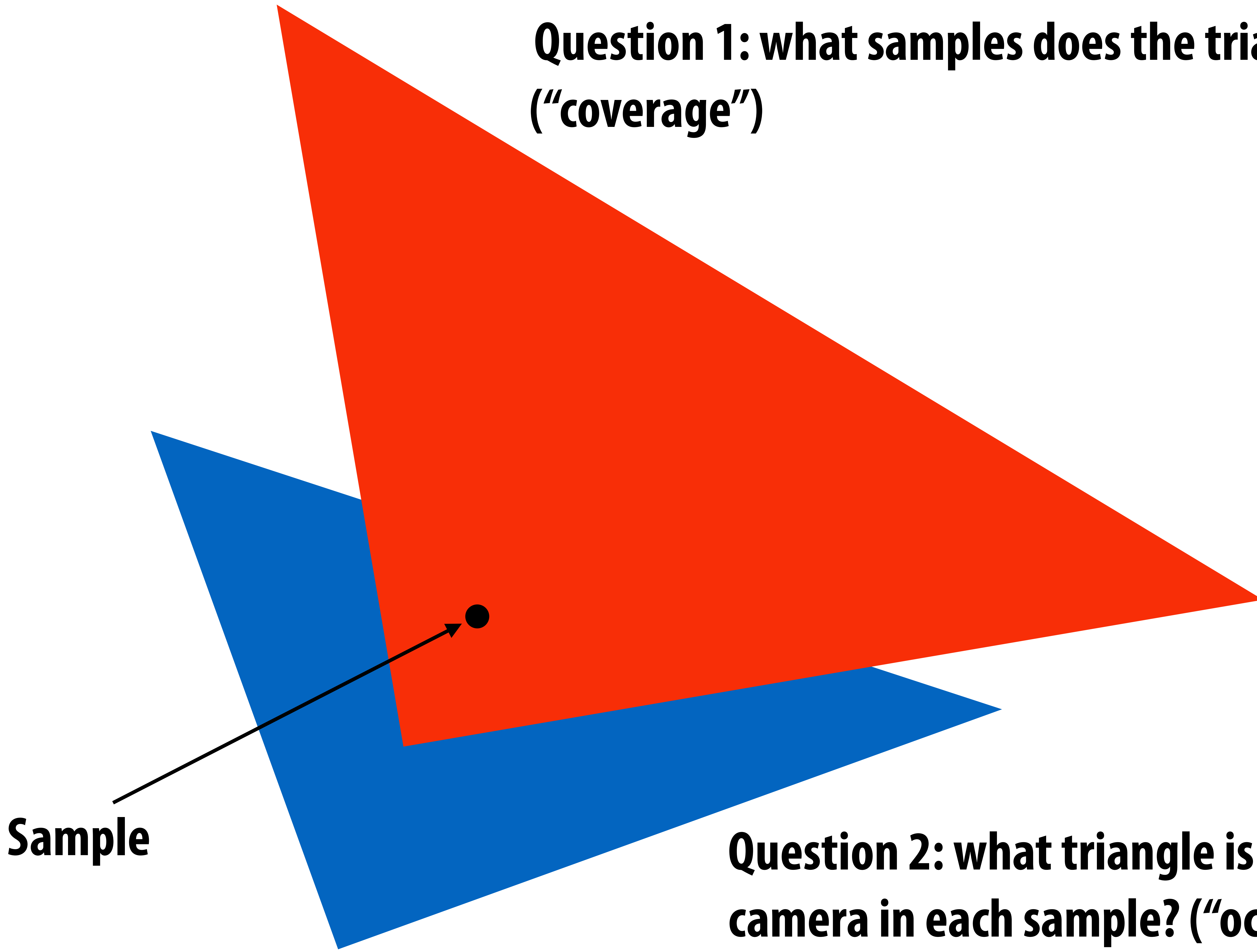
Can we do better? Of course… but you'll have to wait until next class

Rendering via ray casting:
(one common use of ray-scene intersection tests)

Rasterization and ray casting are two algorithms for solving the same problem: determining "visibility from a camera"

Recall triangle visibility:

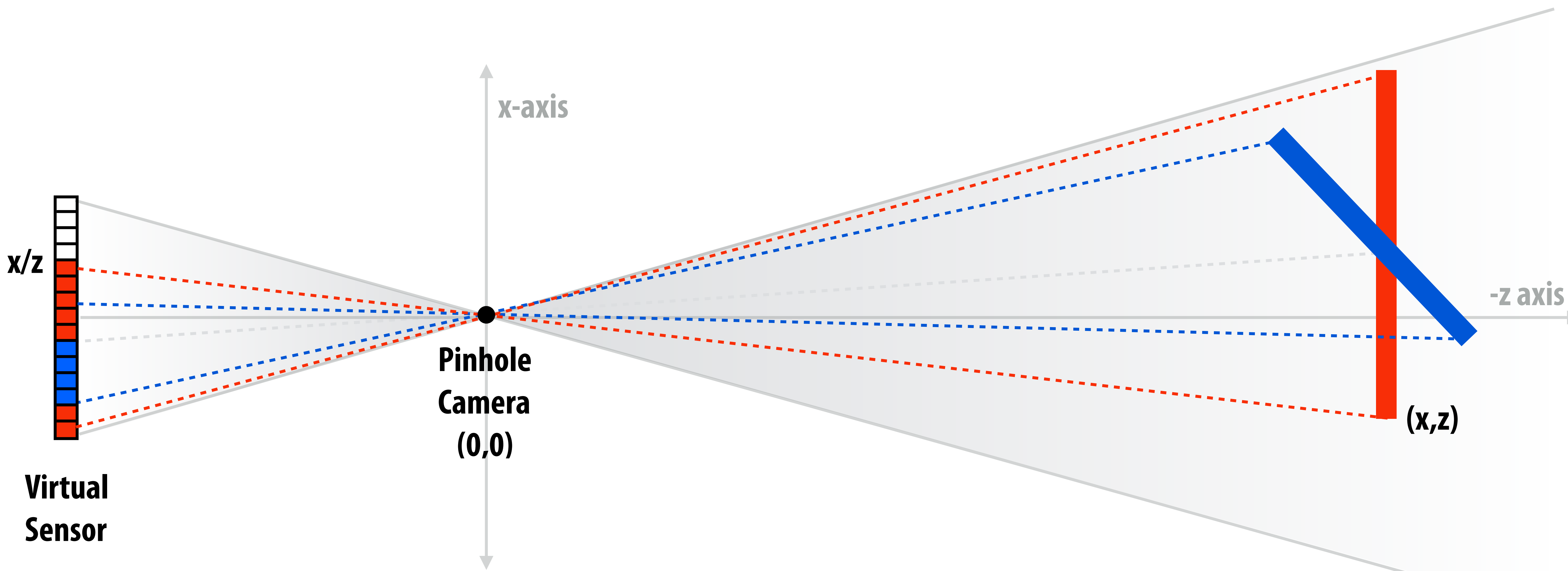
Question 1: what samples does the triangle overlap? ("coverage")



Question 2: what triangle is closest to the camera in each sample? ("occlusion")

The visibility problem

- **What scene geometry is visible at each screen sample?**
  - What scene geometry *projects* onto screen sample points? (coverage)
  - Which geometry is visible from the camera at each sample? (occlusion)



Basic rasterization algorithm

Sample = 2D point

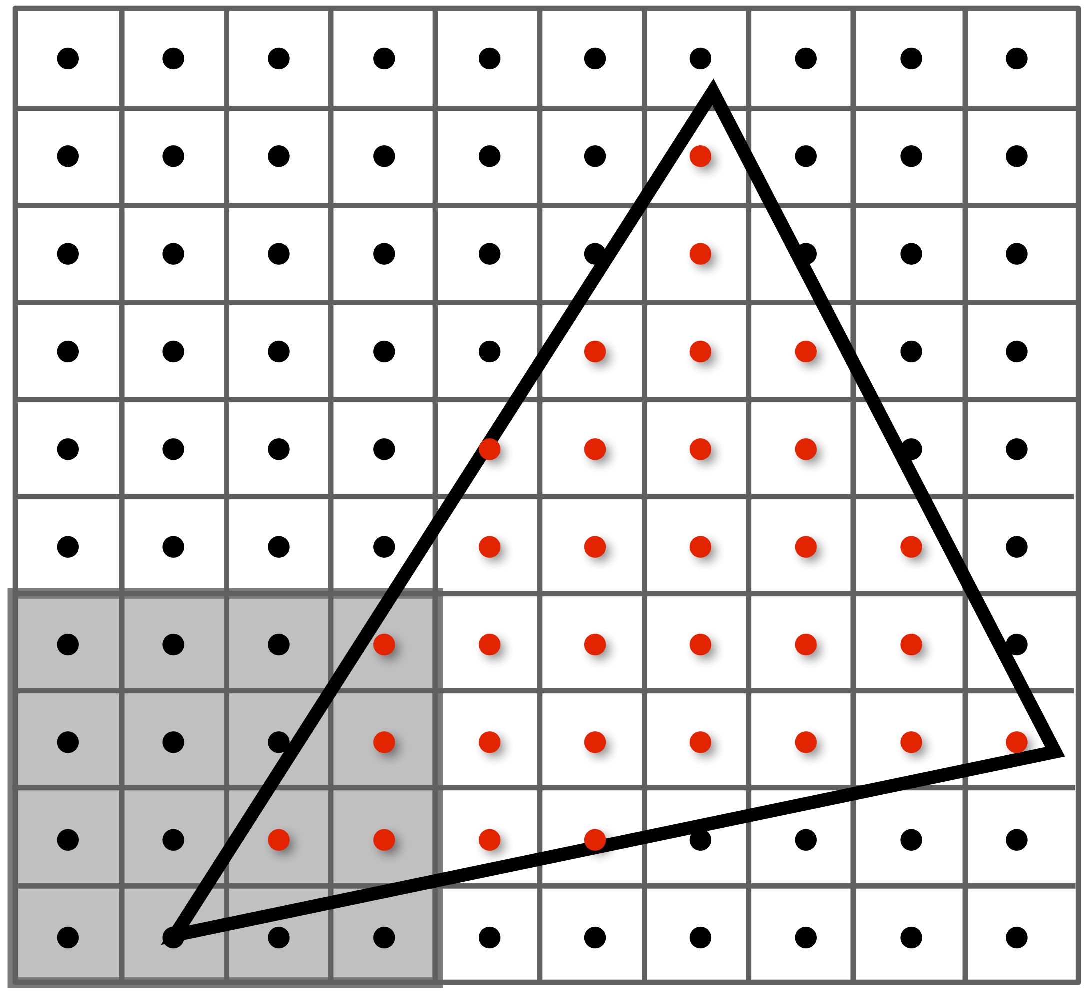
Coverage: 2D triangle/sample tests (does projected triangle cover 2D sample point)

Occlusion: depth buffer

```
initialize z_closest[] to INFINITY               // store closest-surface-so-far for all samples
initialize color[]                               // store scene color for all samples
for each triangle t in scene:                    // loop 1: over triangles
    t_proj = project_triangle(t)
    for each 2D sample s in frame buffer:        // loop 2: over visibility samples
        if (t_proj covers s)
            compute color of triangle at sample
            if (depth of t at s is closer than z_closest[s])
                update z_closest[s] and color[s]
```

"Given a triangle, find the samples it covers"

(finding the samples is relatively easy since they are distributed uniformly on screen)



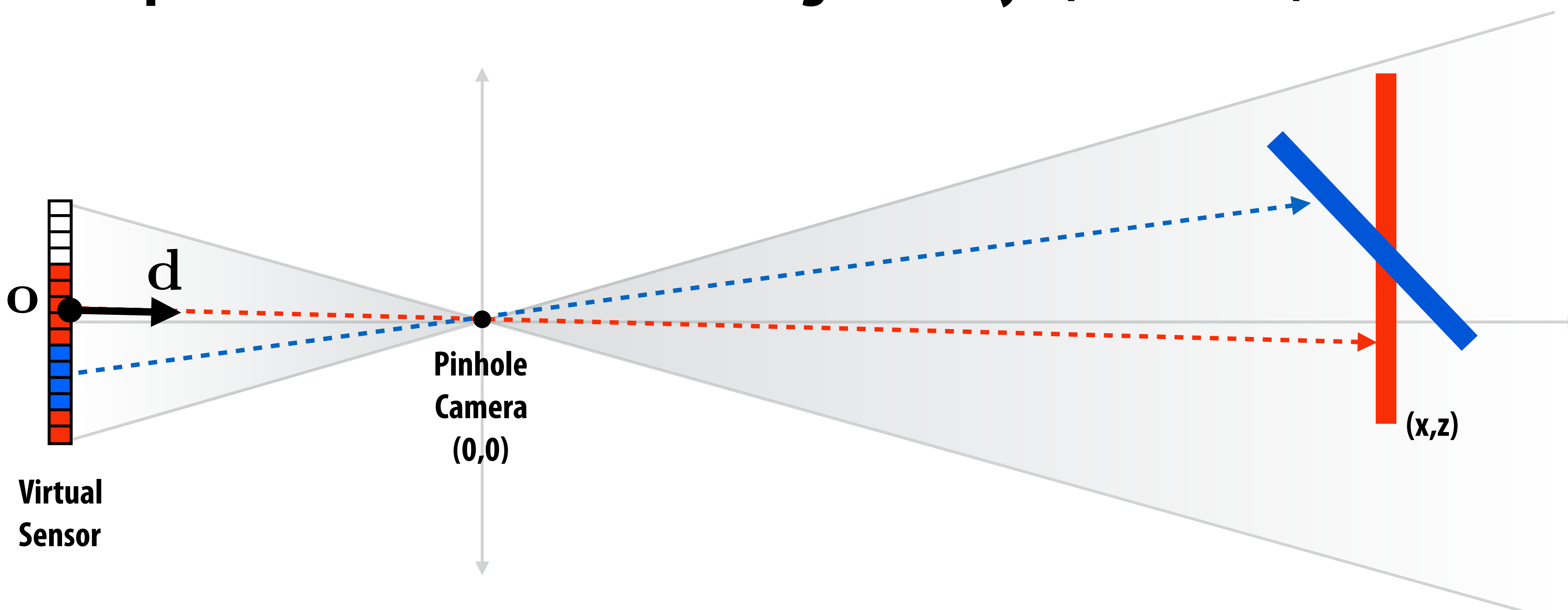
More efficient hierarchical rasterization:

For each TILE of image

If triangle overlaps tile, check all samples in tile

The visibility problem (described differently)

- In terms of casting rays from the camera:
  - Is a scene primitive hit by a ray originating from a point on the virtual sensor and traveling through the aperture of the pinhole camera? (coverage)
  - What primitive is the first hit along that ray? (occlusion)



Basic ray casting algorithm

Sample = a ray in 3D

Coverage: 3D ray-triangle intersection tests (does ray "hit" triangle)

Occlusion: closest intersection along ray

```
initialize color[]                                 // store scene color for all samples
for each sample s in frame buffer:                 // loop 1: over visibility samples (rays)
    r = ray from s on sensor through pinhole aperture
    r.min_t = INFINITY                             // only store closest-so-far for current ray
    r.tri = NULL;
    for each triangle tri in scene:                 // loop 2: over triangles
        if (intersects(r, tri)) {                   // 3D ray-triangle intersection test
            if (intersection distance along ray is closer than r.min_t)
                update r.min_t and r.tri = tri;
        }
    color[s] = compute surface color of triangle r.tri at hit point
```

Compared to rasterization approach: just a reordering of the loops!

"Given a ray, find the closest triangle it hits."

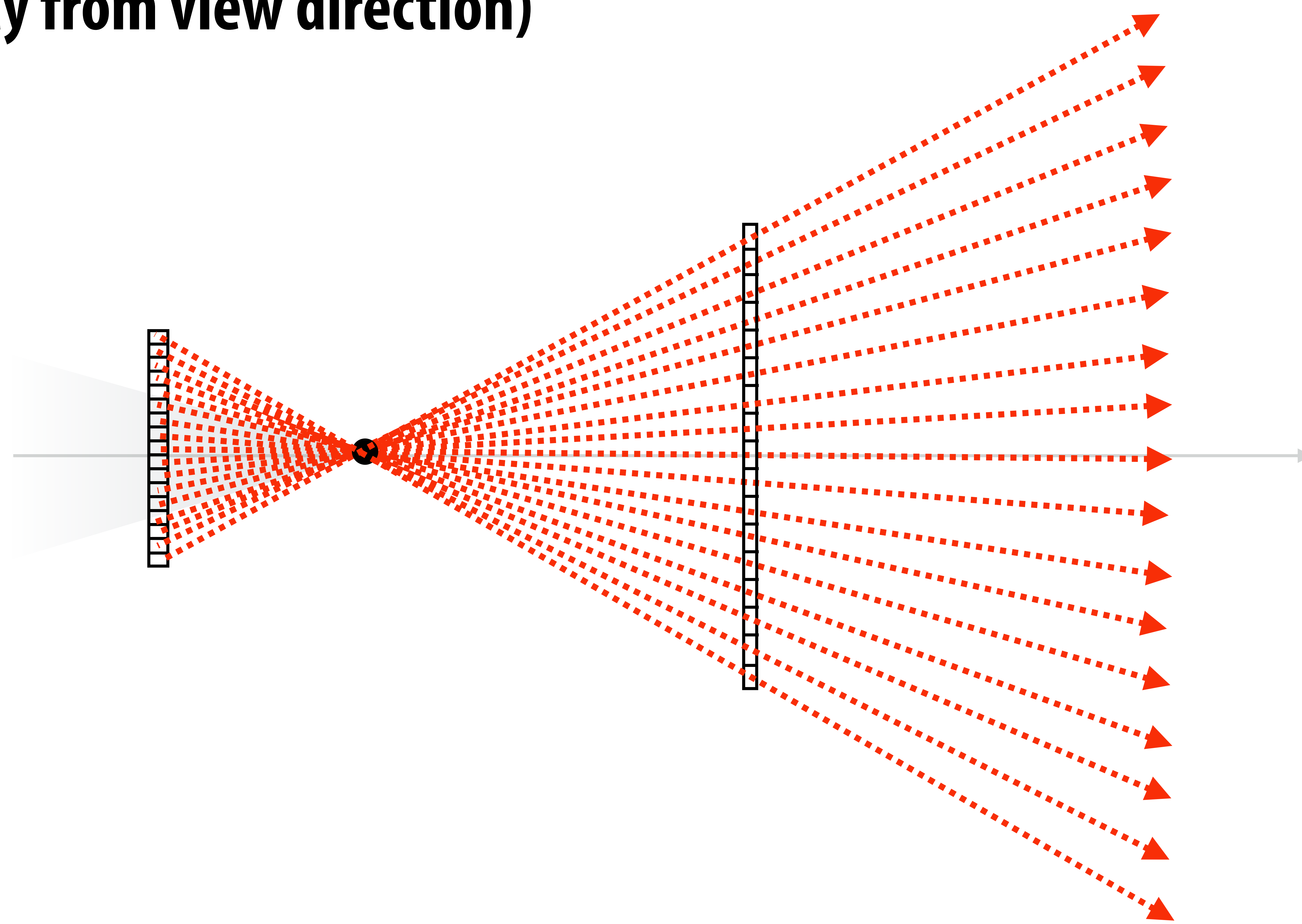
Basic rasterization vs. ray casting

- **Rasterization:**
  - Proceeds in triangle order (for all triangles)
  - Store entire depth buffer (requires access to 2D array of fixed size)
  - Do not have to store entire scene geometry in memory
    - Naturally supports unbounded size scenes

- **Ray casting:**
  - Proceeds in screen sample order (for all rays)
    - Do not have to store closest depth so far for the entire screen (just the current ray)
    - This is the natural order for rendering transparent surfaces (process surfaces in the order the are encountered along the ray: front-to-back)
  - Must store entire scene geometry for fast access

In other words…

- Rasterization is a efficient implementation of ray casting where:
  - Ray-scene intersection is computed for a batch of rays
  - All rays in the batch originate from same origin
  - Rays are distributed uniformly in plane of projection
  (Note: not uniform distribution in angle… angle between rays is smaller away from view direction)

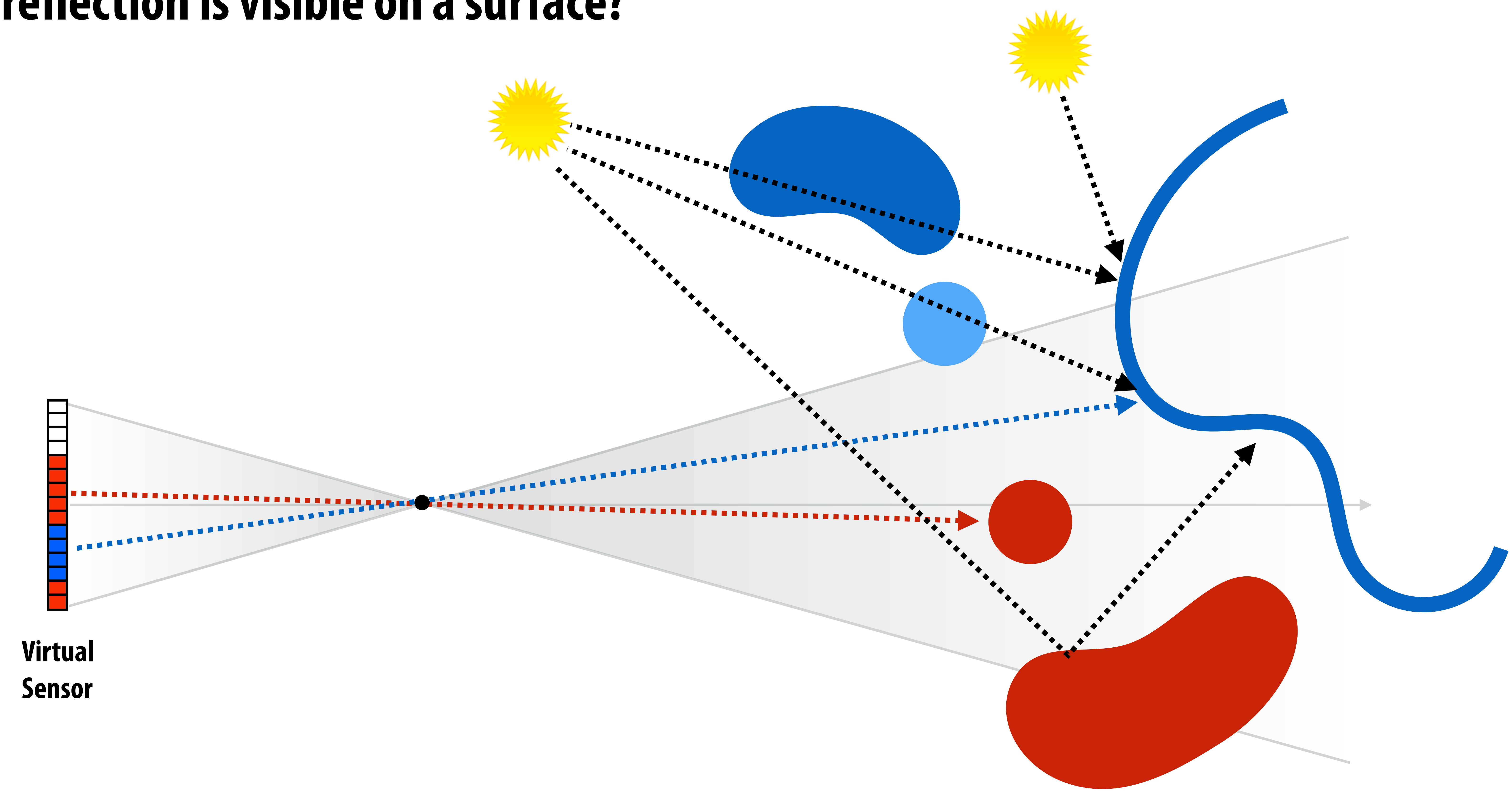


Generality of ray-scene queries

What object is visible to the camera?

What light sources are visible from a point on a surface (is a surface in shadow?)

What reflection is visible on a surface?



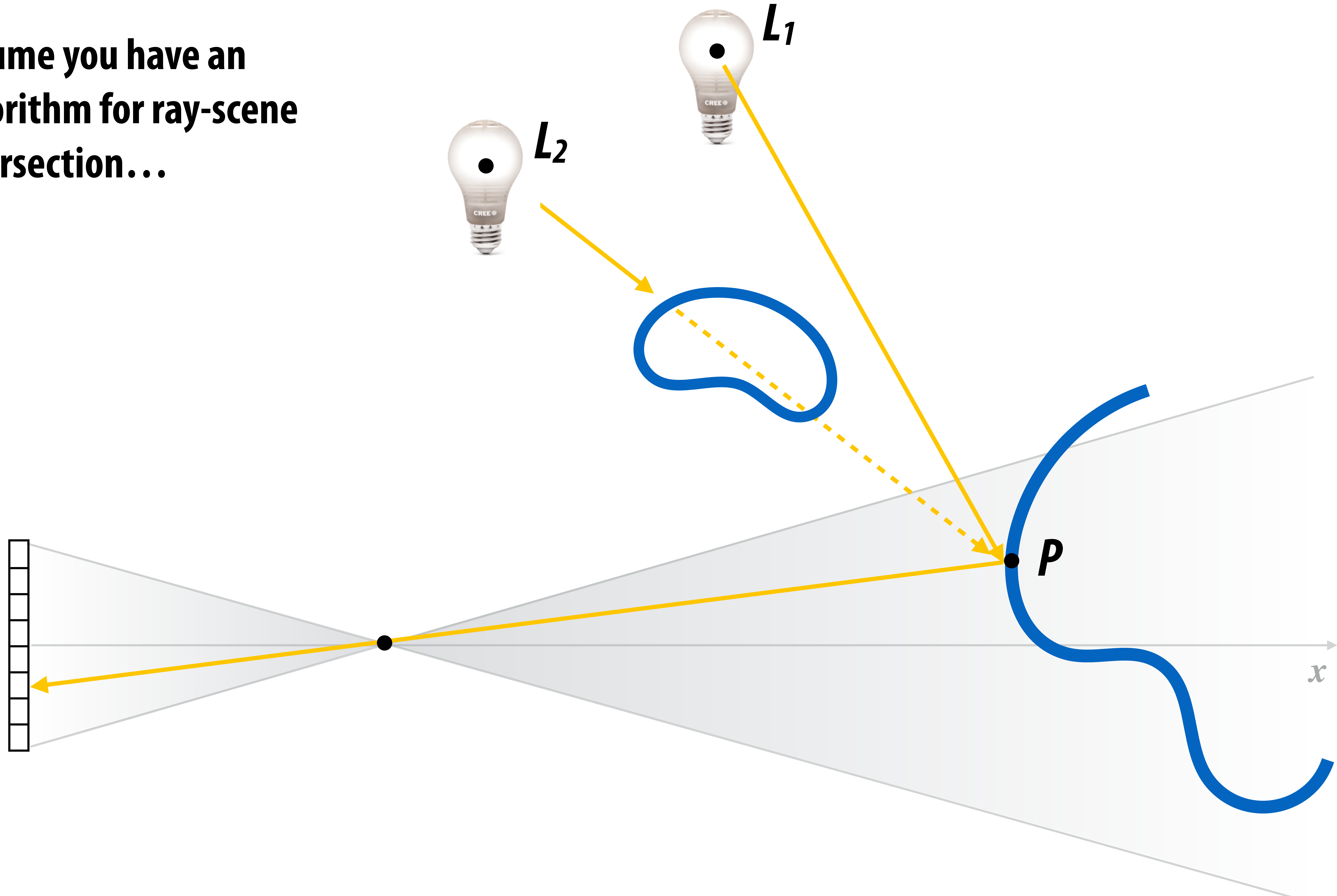
In contrast, rasterization is a highly-specialized solution for computing visibility for a set of uniformly distributed rays originating from the same point (most often: the camera)

Shadows

Image credit: Grand Theft Auto V

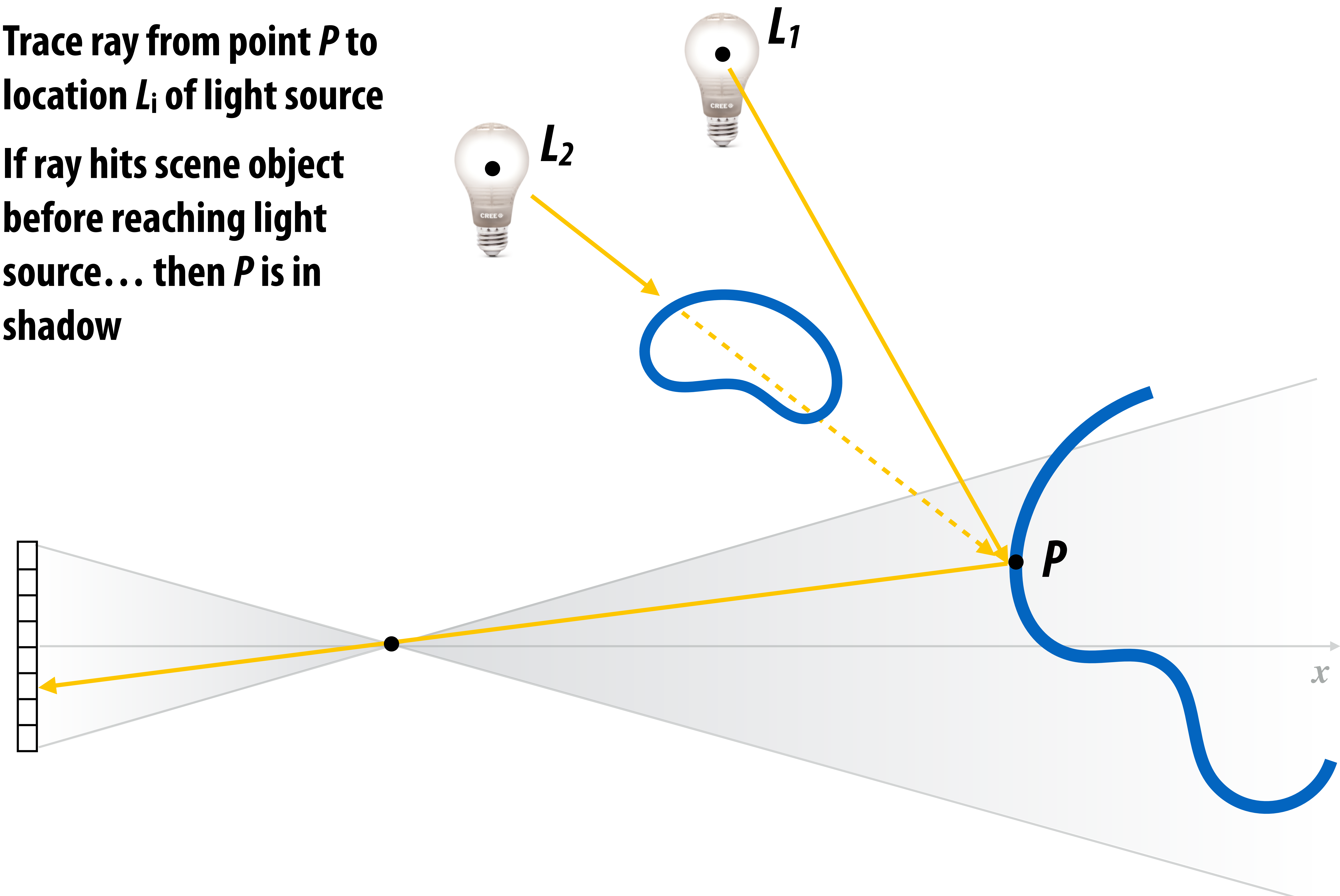
How to compute if a surface point is in shadow?

Assume you have an algorithm for ray-scene intersection…



A simple shadow computation algorithm

- Trace ray from point *P* to location L_i of light source
- If ray hits scene object before reaching light source… then *P* is in shadow



Direct illumination + reflection + transparency

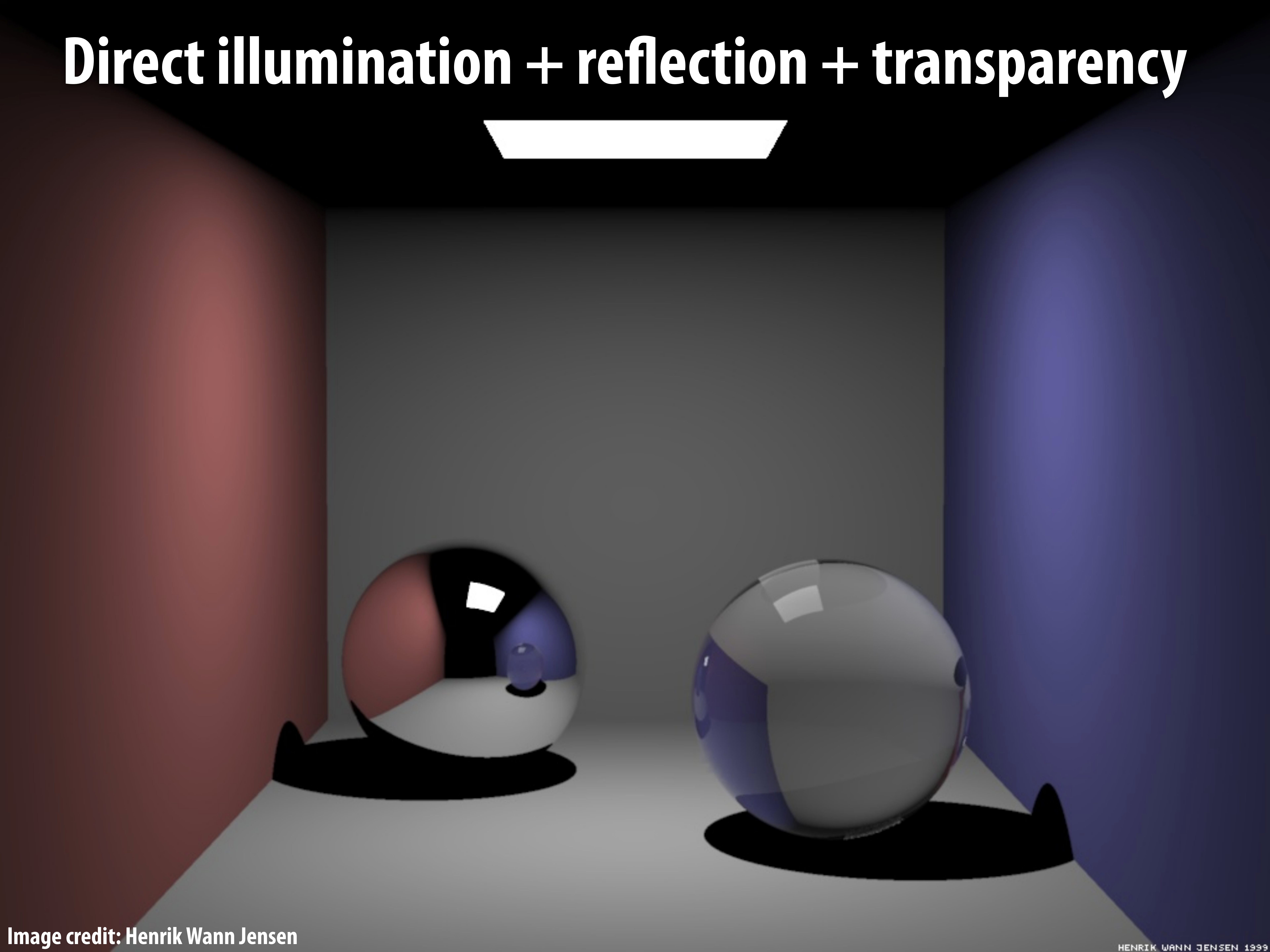


Image credit: Henrik Wann Jensen

Global illumination solution

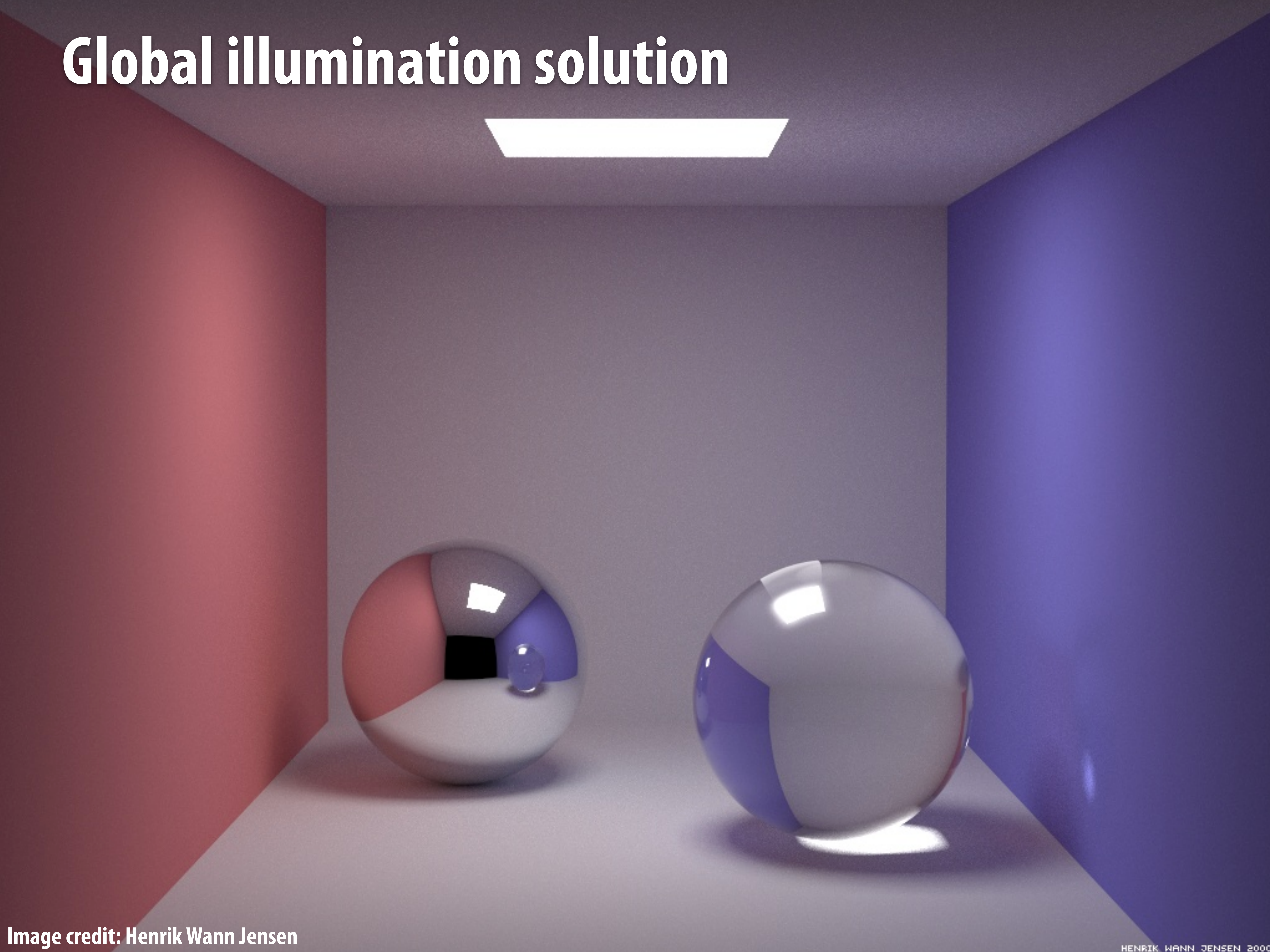


Image credit: Henrik Wann Jensen

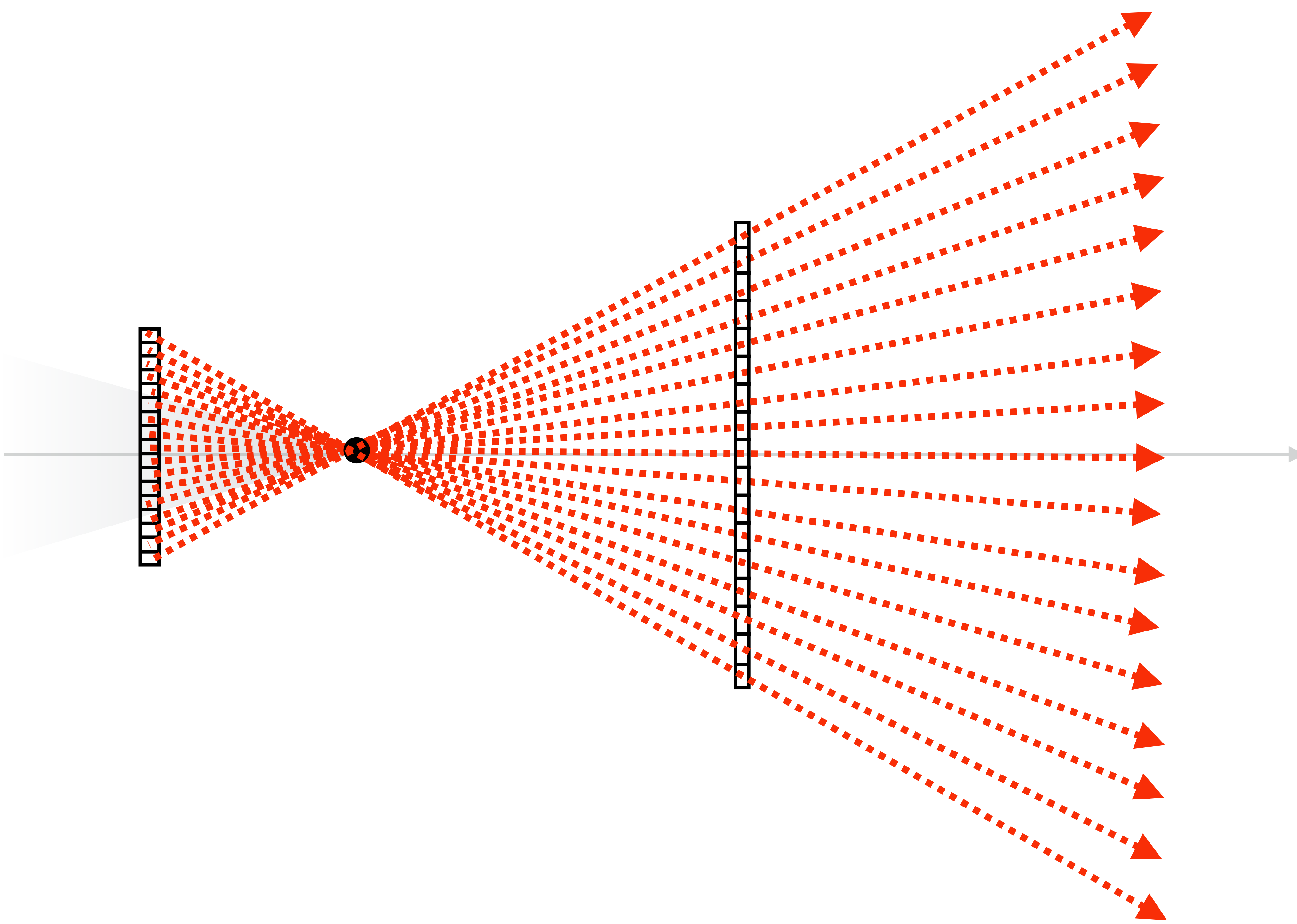
Direct illumination

Sixteen-bounce global illumination

Recall rasterization / ray casting relationship

- Rasterization is a efficient implementation of ray casting where:
  - Ray-scene intersection is computed for a batch of rays
  - All rays in the batch originate from same origin
  - Rays are distributed uniformly in plane of projection

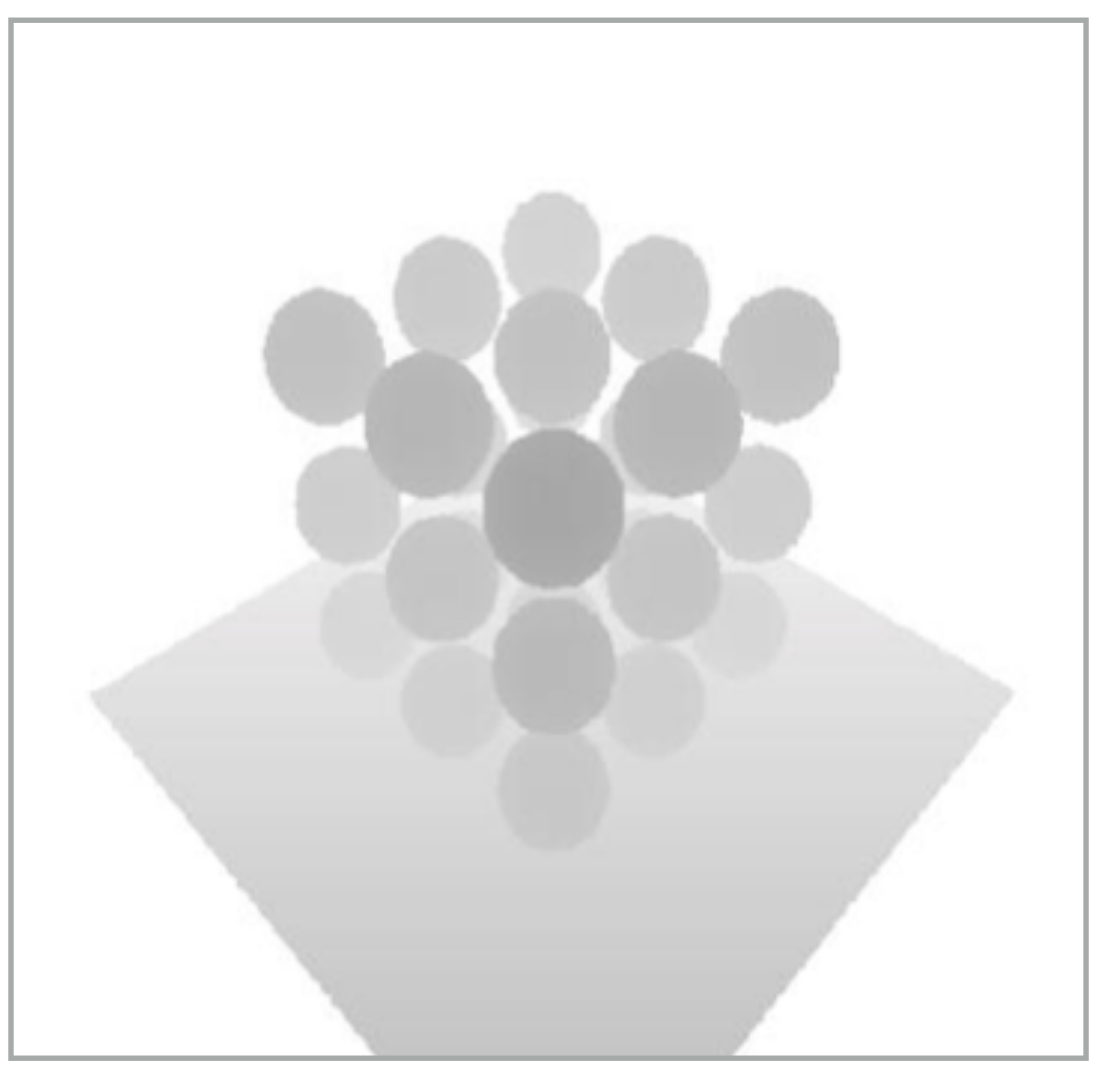
  (Note: not uniform distribution in angle… angle between rays is smaller away from view direction)



Shadow mapping: ray origin for rasterization need not be the scene's camera position [Williams 78]

1. Place camera at position of a point light source
2. Render scene to compute depth to closest object to light along uniformly distributed "shadow rays" (answer stored in depth buffer)
3. Store precomputed shadow ray intersection results in a texture

"Shadow map" = depth map from perspective of a point light.
(Stores closest intersection along each shadow ray in a texture)



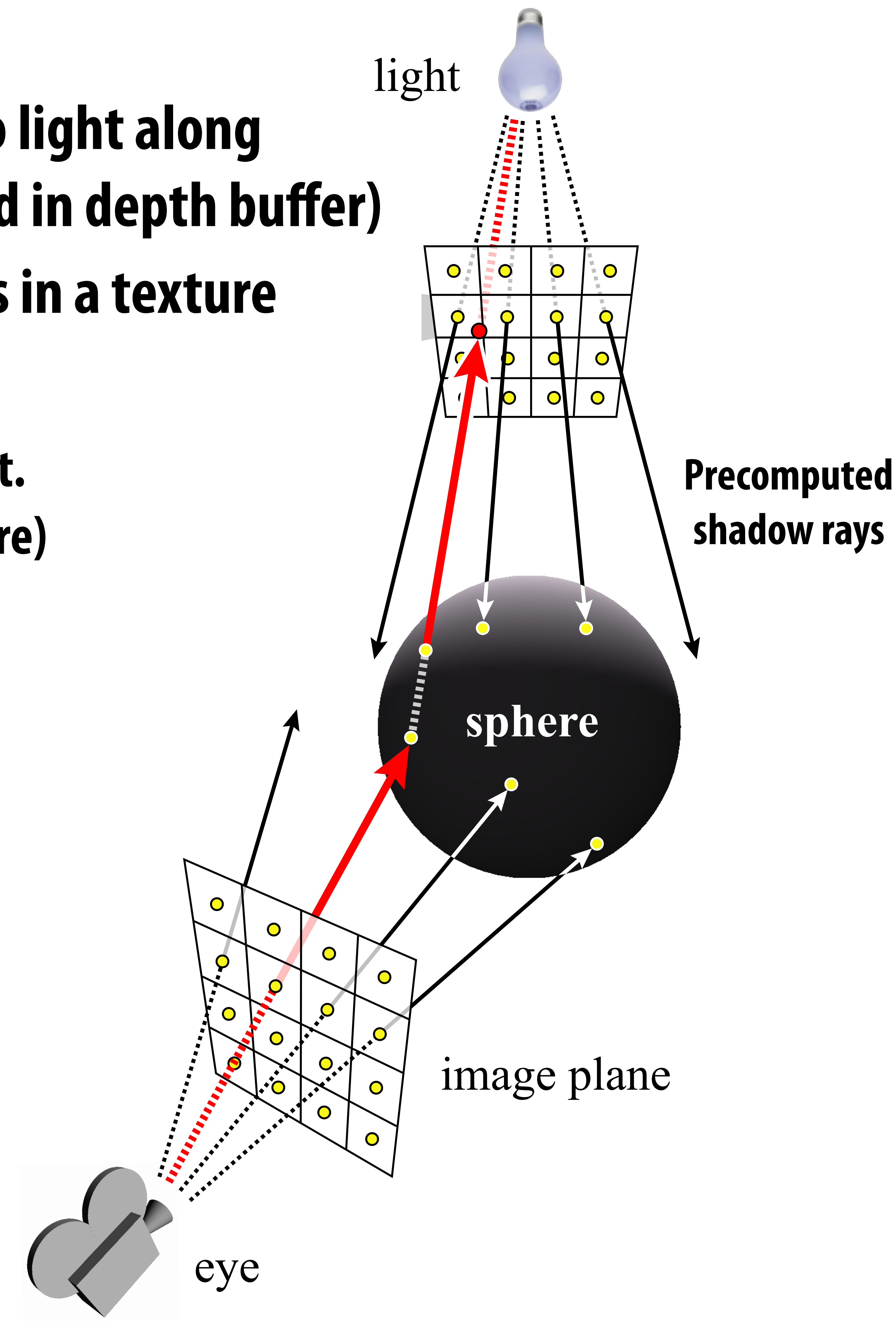
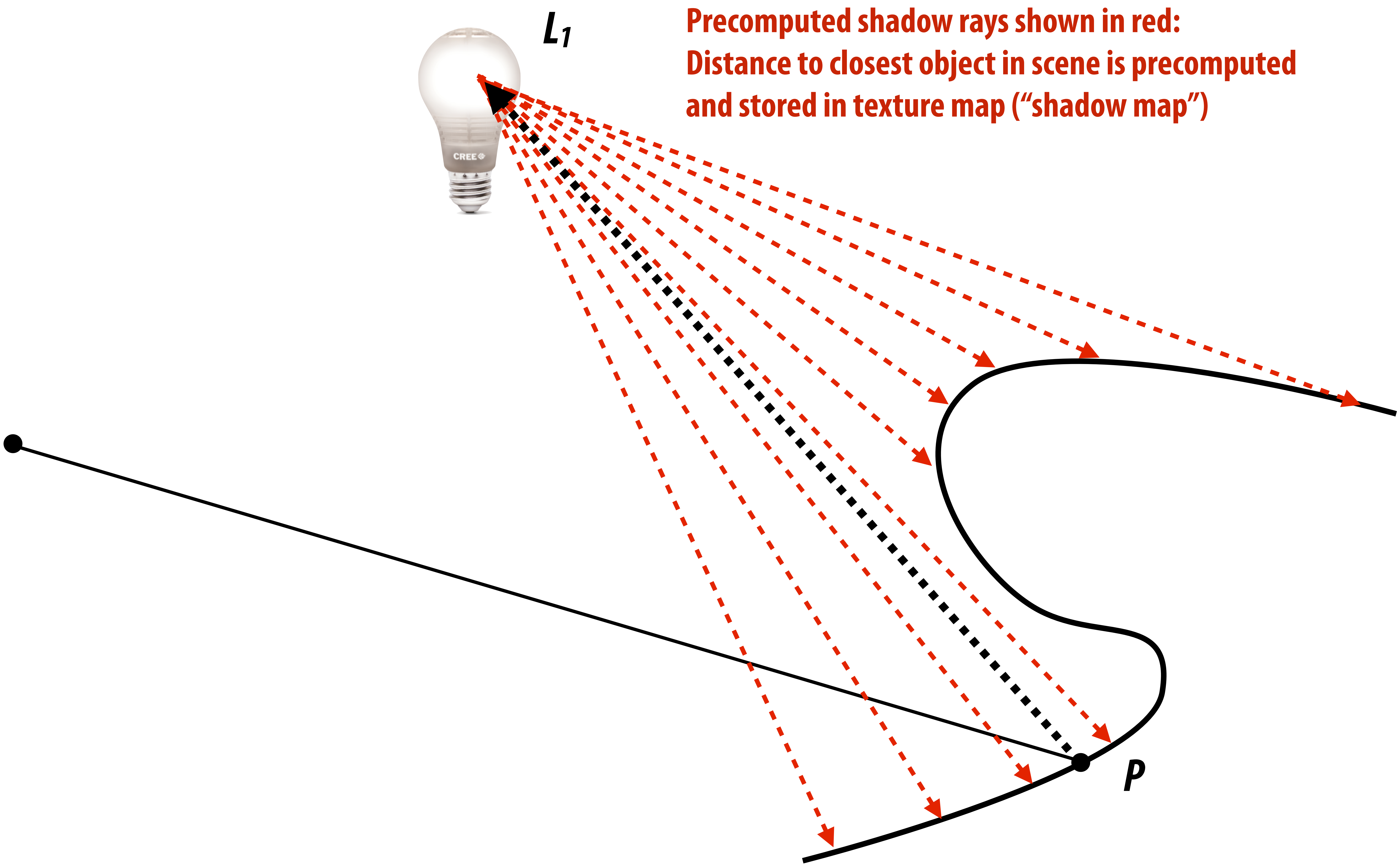


Image credits: Segal et al. 92, NVIDIA

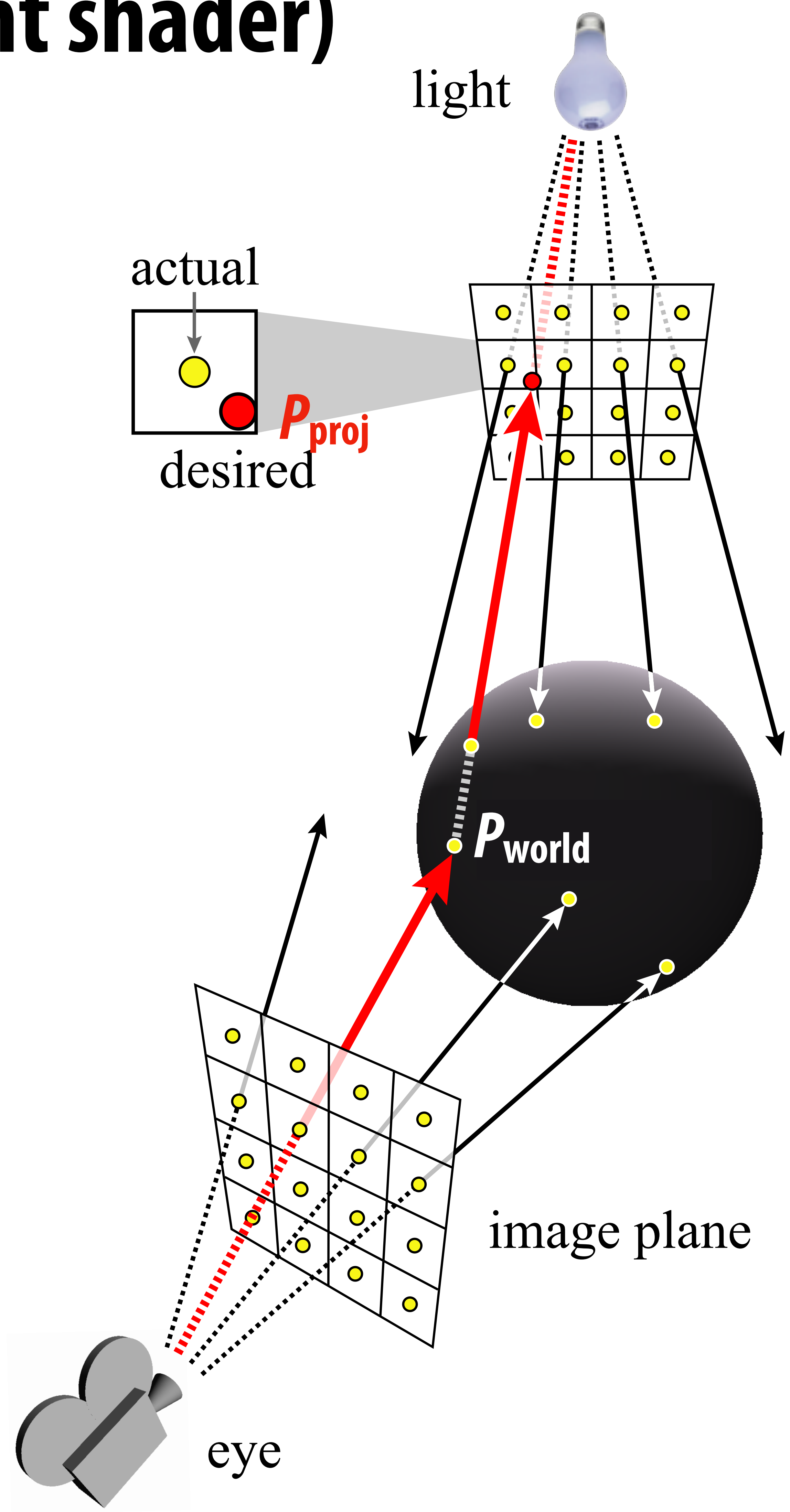
Shadow texture lookup approximates visibility result when shading fragment at *P*



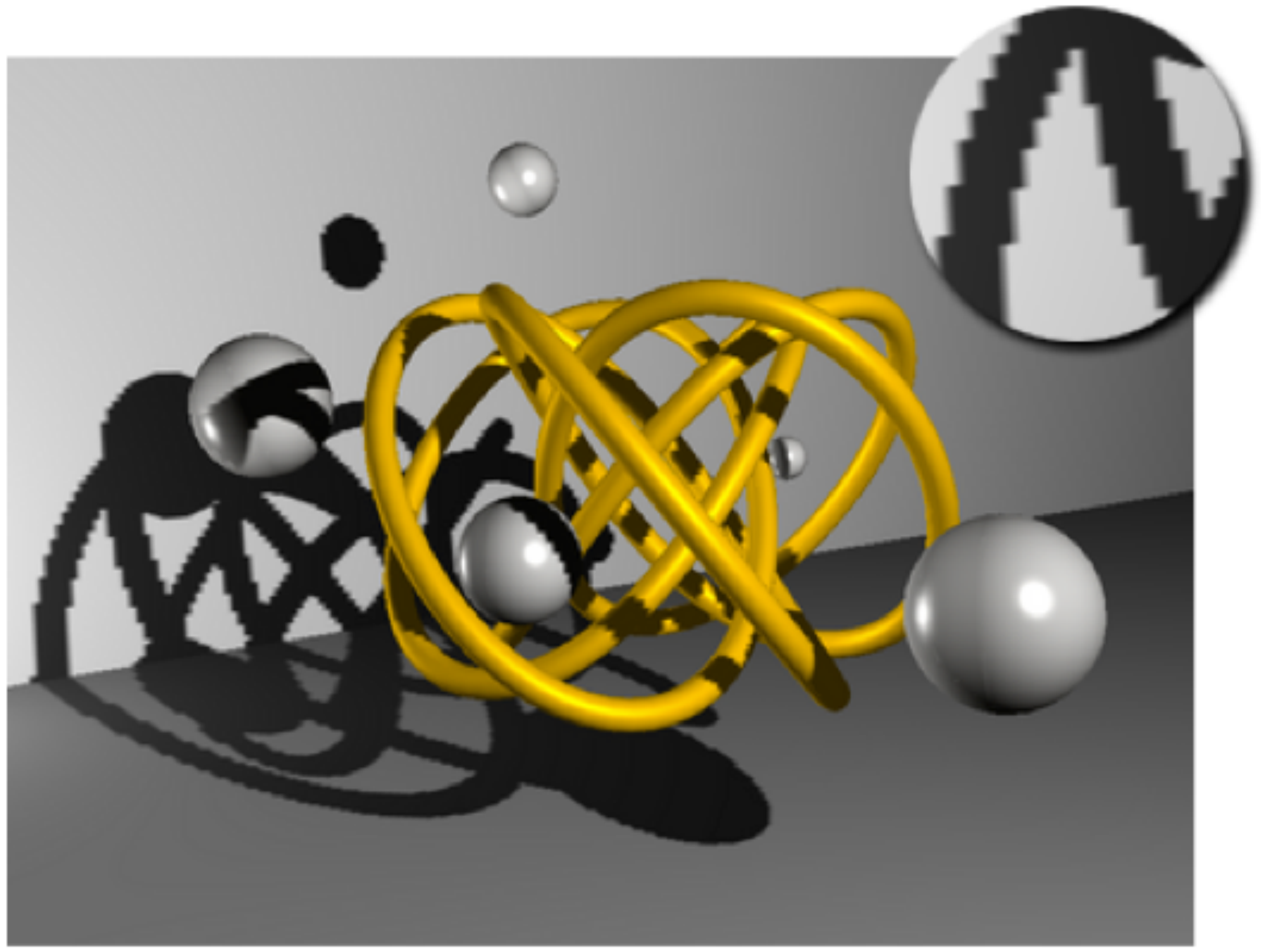
Shadow mapping pseudocode

(this logic would be implemented in fragment shader)

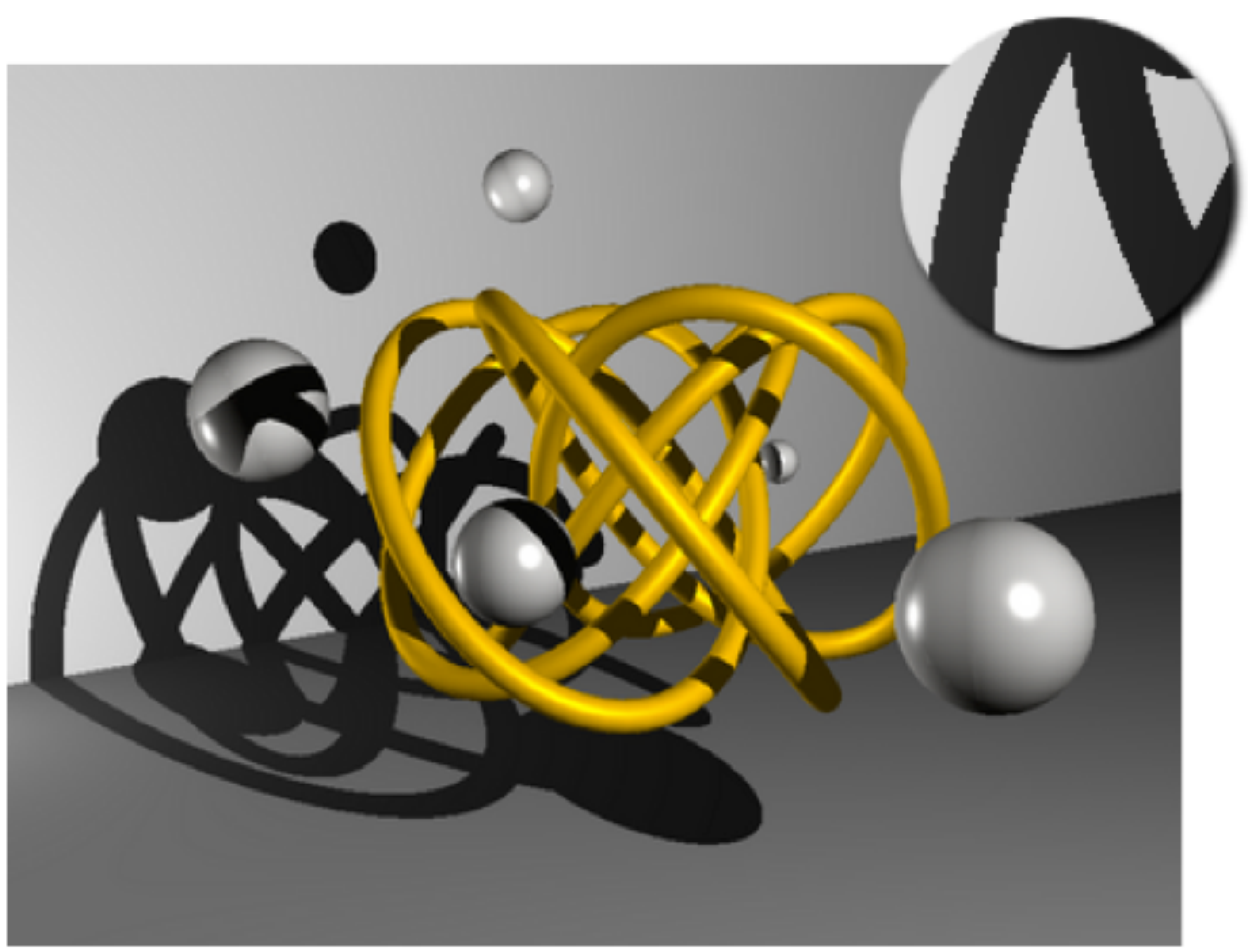
- Given world-space point P_{world}, light position (L), *and* light direction (D)
- Transform P into "light space", defined by light position at origin and $-Z$ aligned with D
- Project transformed P into P_{proj}
- Lookup value in shadow map at (P_{proj}.x, P_{proj}.y)
- If value from shadow map is less than |L-P|, then point P is in shadow



Shadow aliasing due to shadow map undersampling



Shadows computed using shadow map

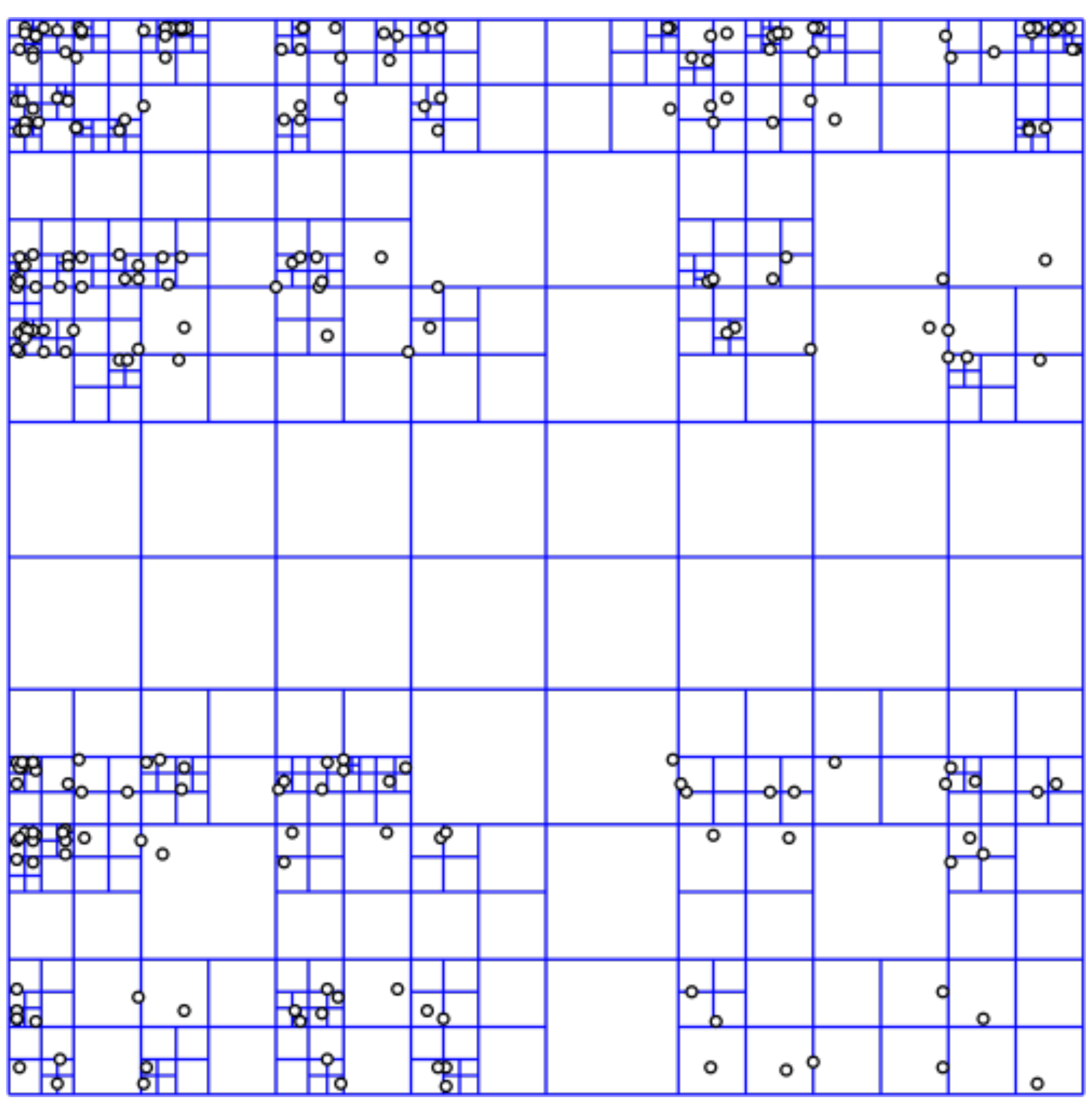


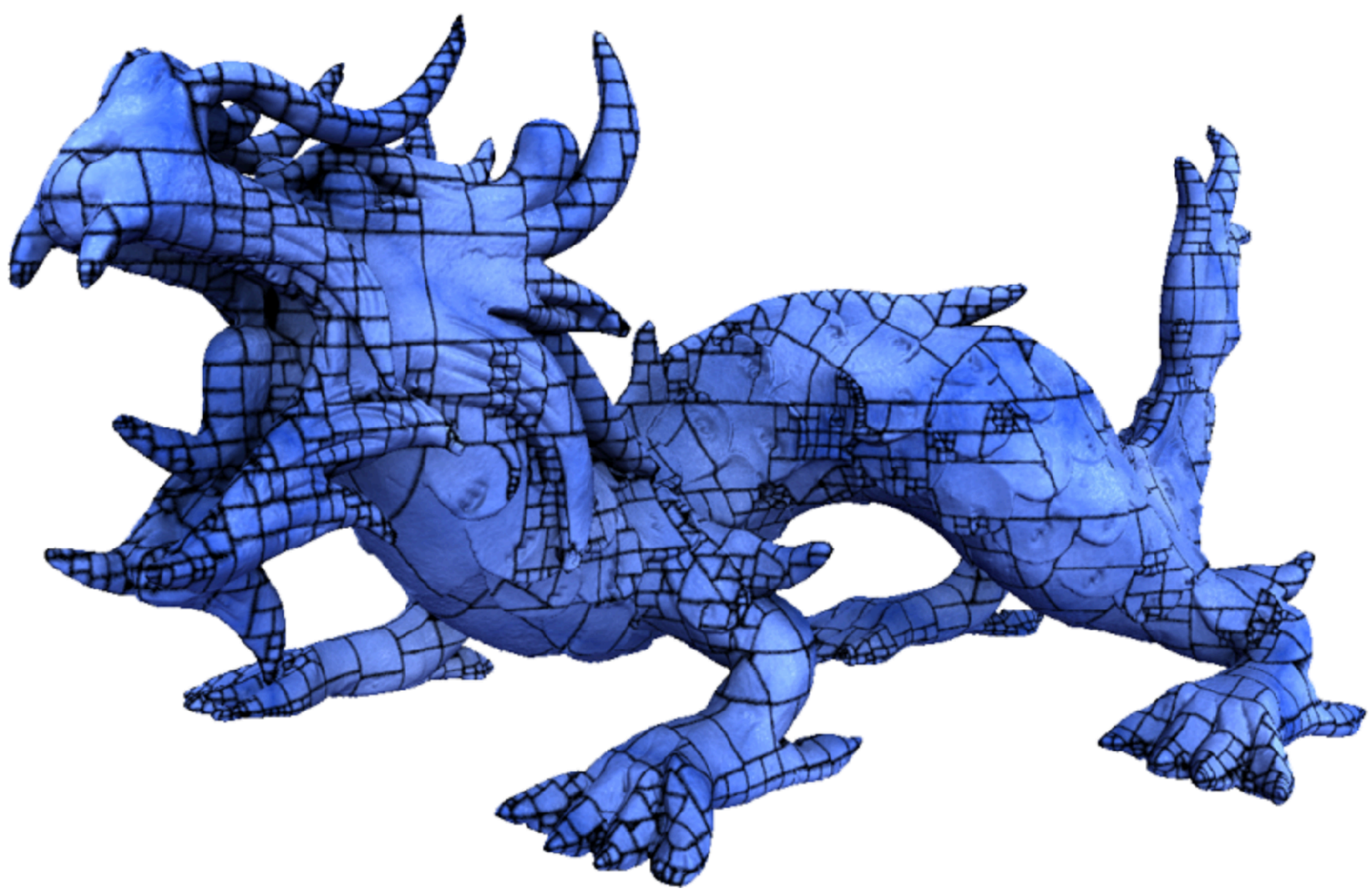
Correct hard shadows (result from computing shadow directly using ray tracing)

Image credit: Johnson et al. TOG 2005

Next time: spatial acceleration data structures

- Testing every primitive in scene to find ray-scene intersection is *slow!*
- Consider linearly scanning through a list vs. binary search
  - can apply this same kind of thinking to geometric queries





Acknowledgements

- Thanks to Keenan Crane for presentation resources