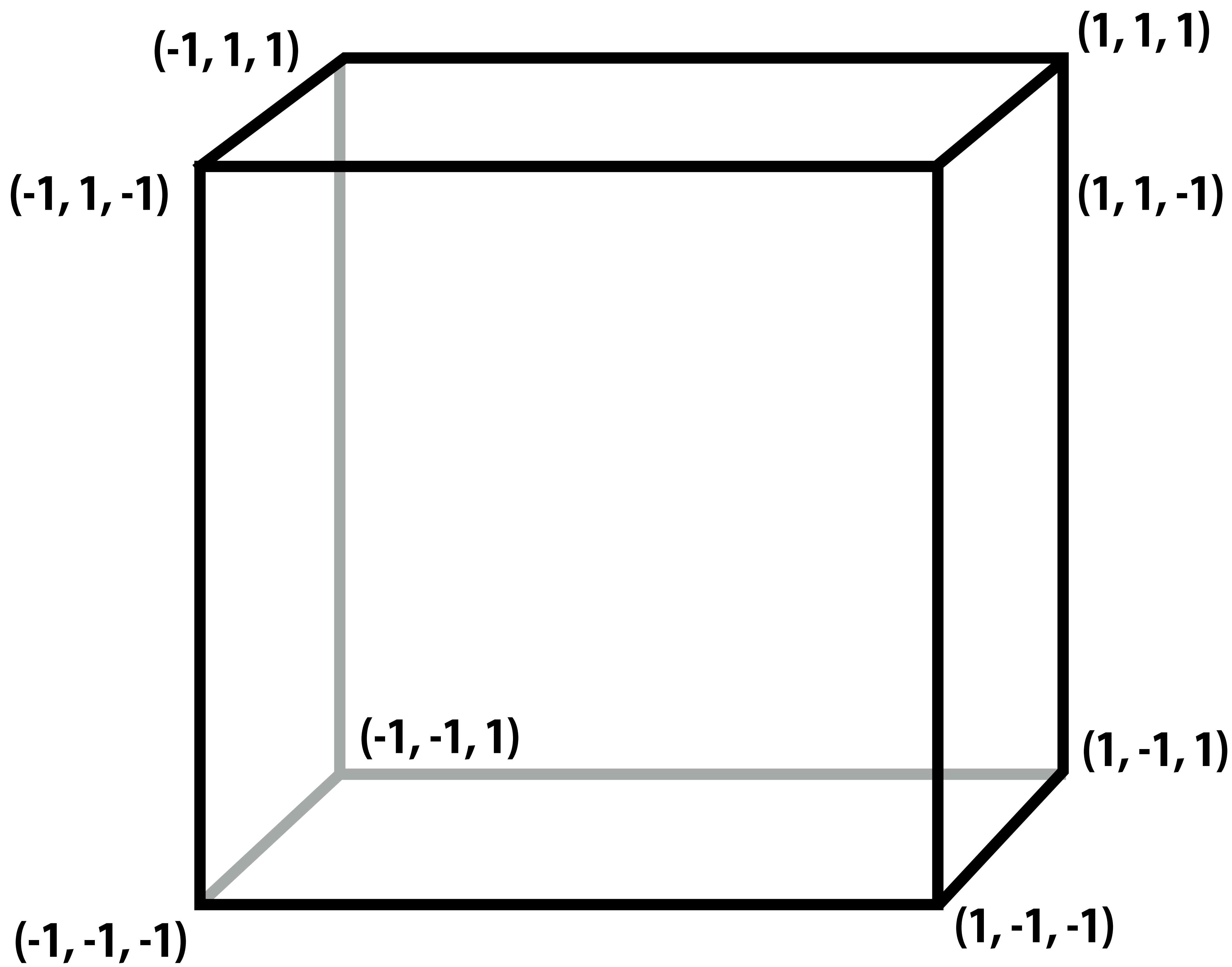
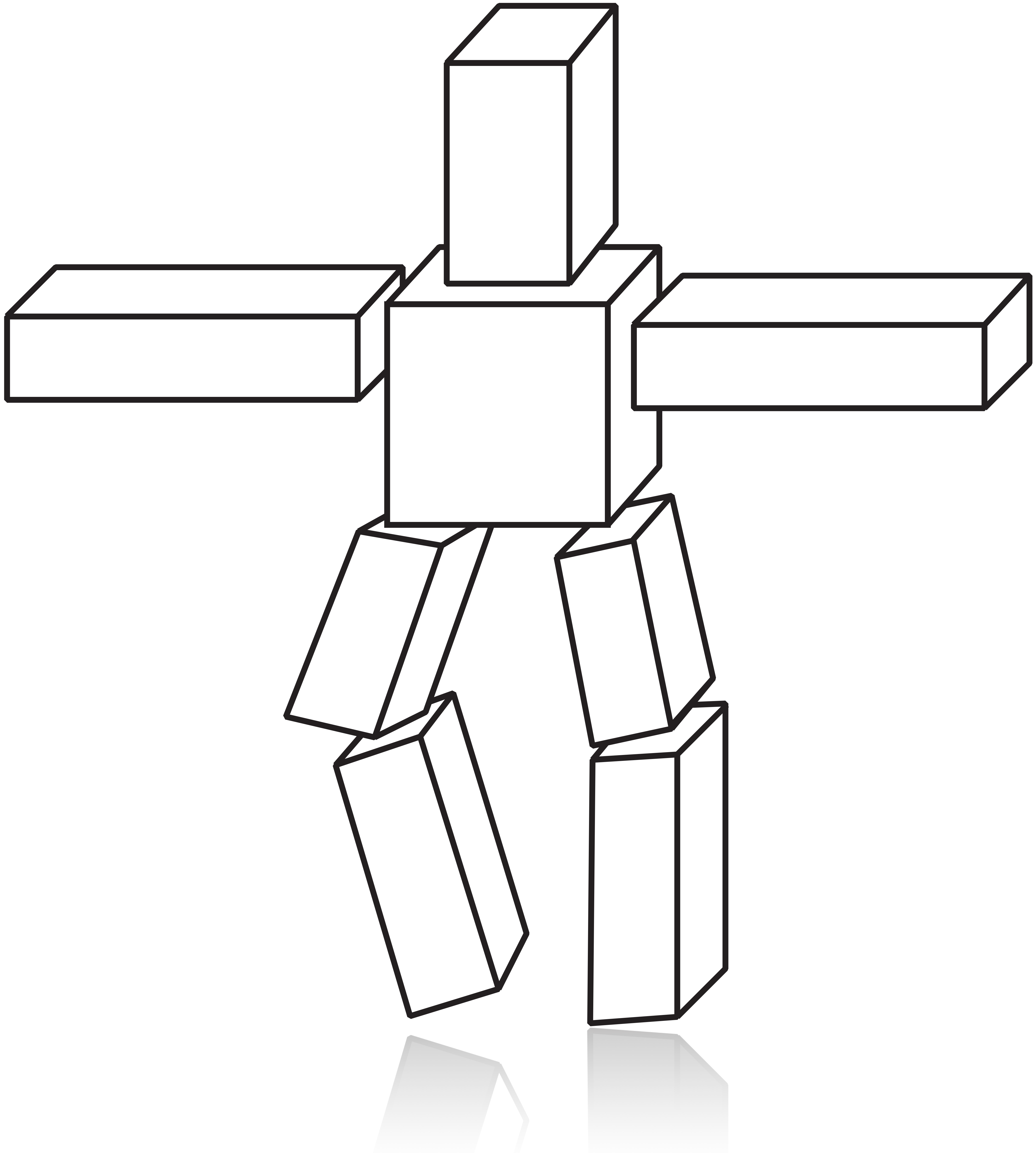
Coordinate Spaces and Transformations

Interactive Computer Graphics
Stanford CS248, Winter 2019

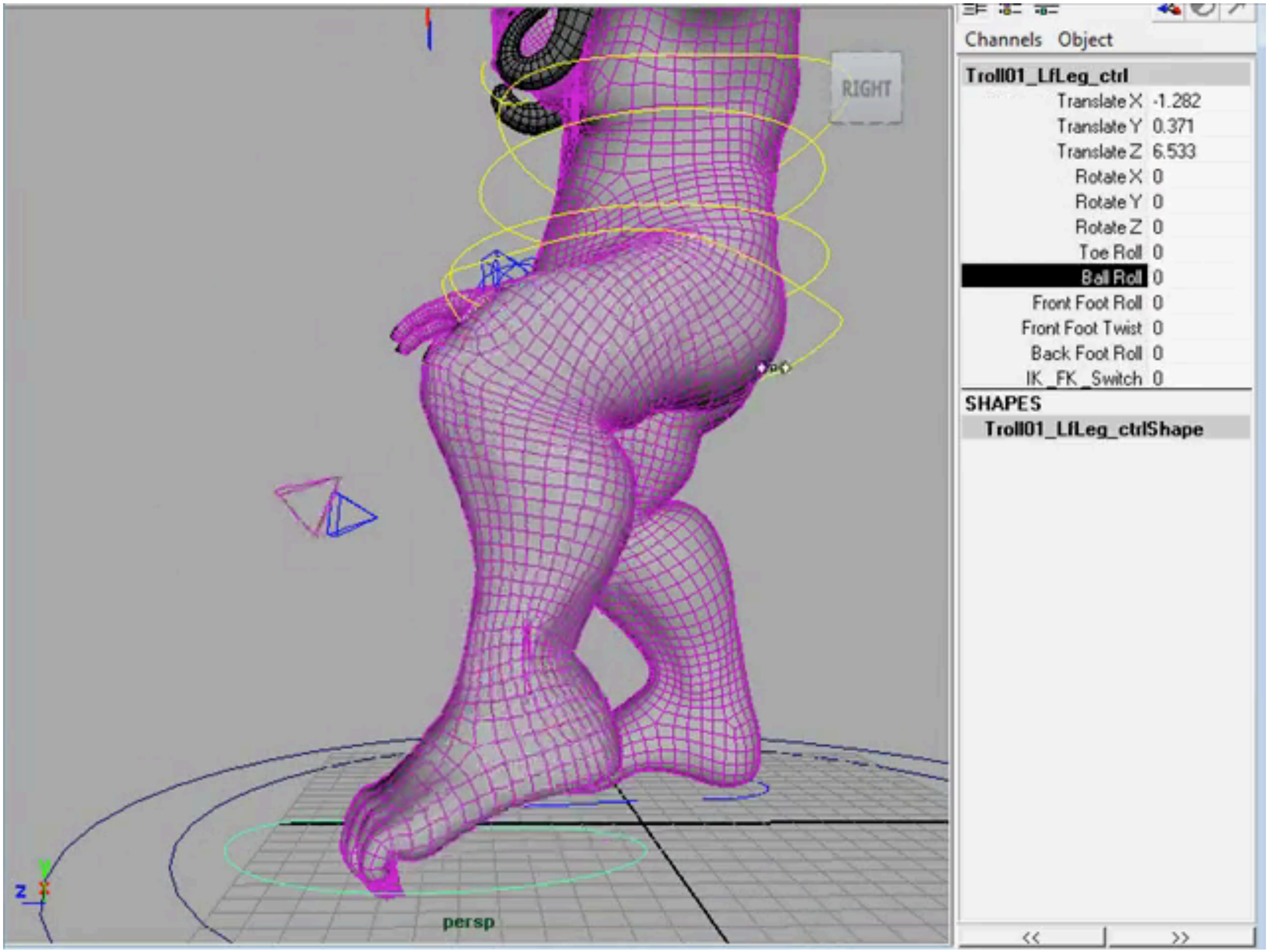
Cube



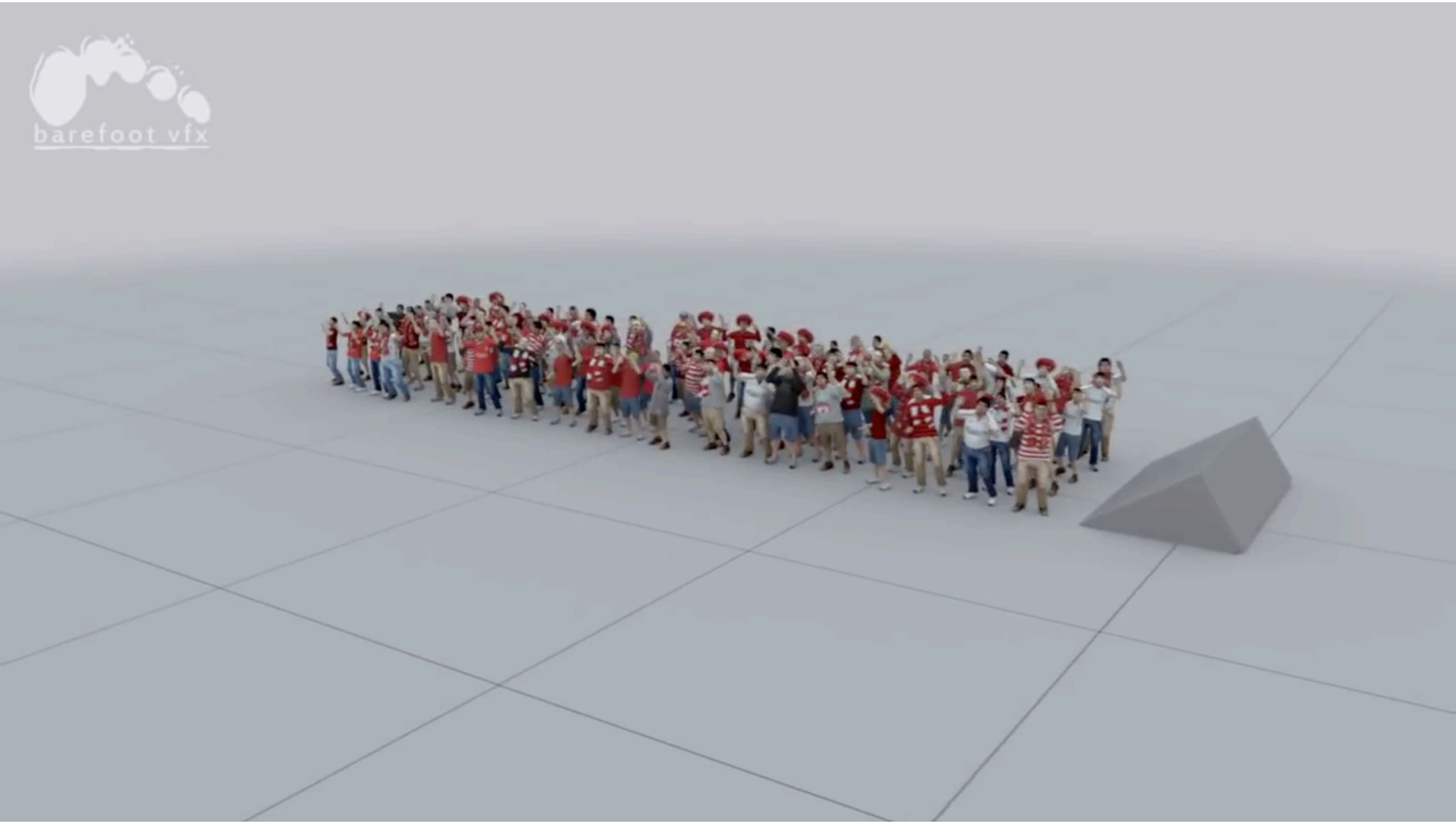
Consider drawing a cube person



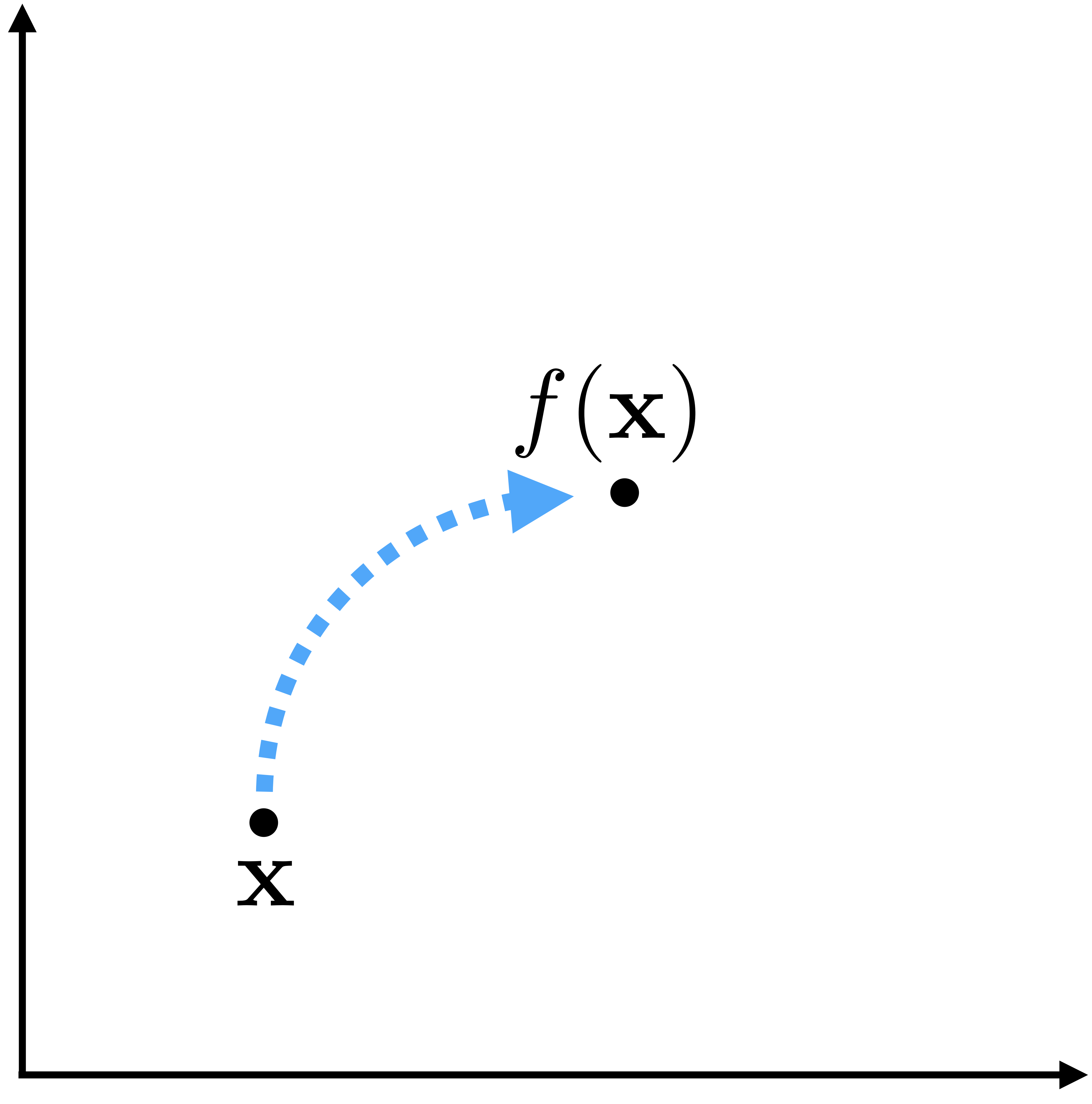
Transformations in character rigging



Transformations in instancing



Basic idea: f transforms $\mathbf{x}$ to $f(\mathbf{x})$



What can we do with *linear* transformations?

- **What does *linear* mean?**

$$f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$$

$$f(a\mathbf{x}) = af(\mathbf{x})$$

- **Cheap to compute**
- **Composition of linear transformations is linear**
  - **Leads to uniform representation of transformations**

Linear transformation

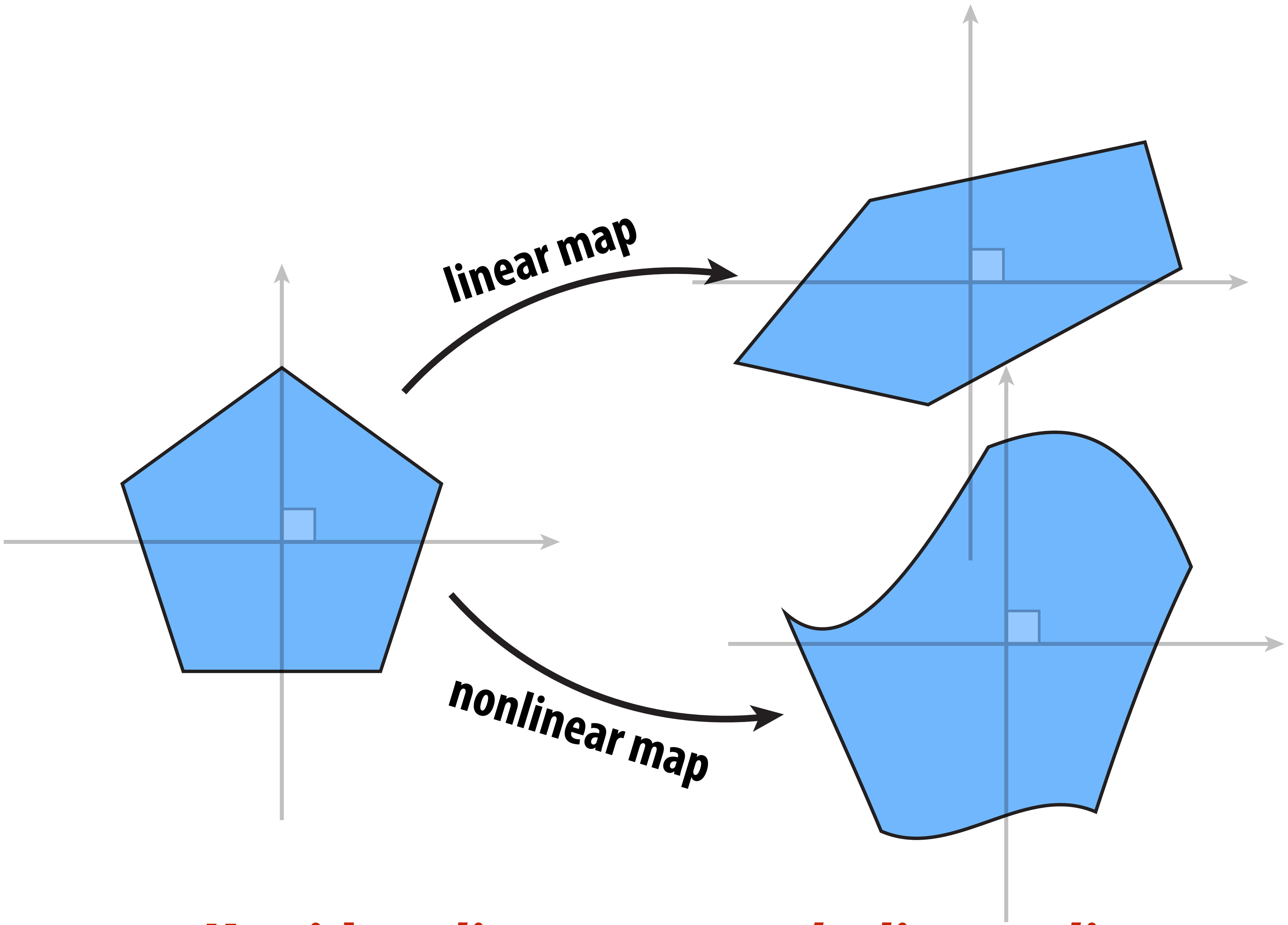
$$f(\mathbf{u}+\mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v})$$
$$f(a\mathbf{u}) = af(\mathbf{u})$$

- **In other words: if it doesn't matter whether we add the vectors and then apply the map, or apply the map and then add the vectors (and likewise for scaling):**

$$\begin{tikzcd}
\mathbf{x},\mathbf{y} \arrow[r, "\text{add first}"] \arrow[d, "\text{apply f first}"] & \mathbf{x}+\mathbf{y} \arrow[d, "\text{then apply f}"] \\
f(\mathbf{x}), f(\mathbf{y}) \arrow[r, "\text{then add}"] & \begin{matrix} f(\mathbf{x})+f(\mathbf{y}) \\ = \\ f(\mathbf{x}+\mathbf{y}) \end{matrix}
\end{tikzcd}$$

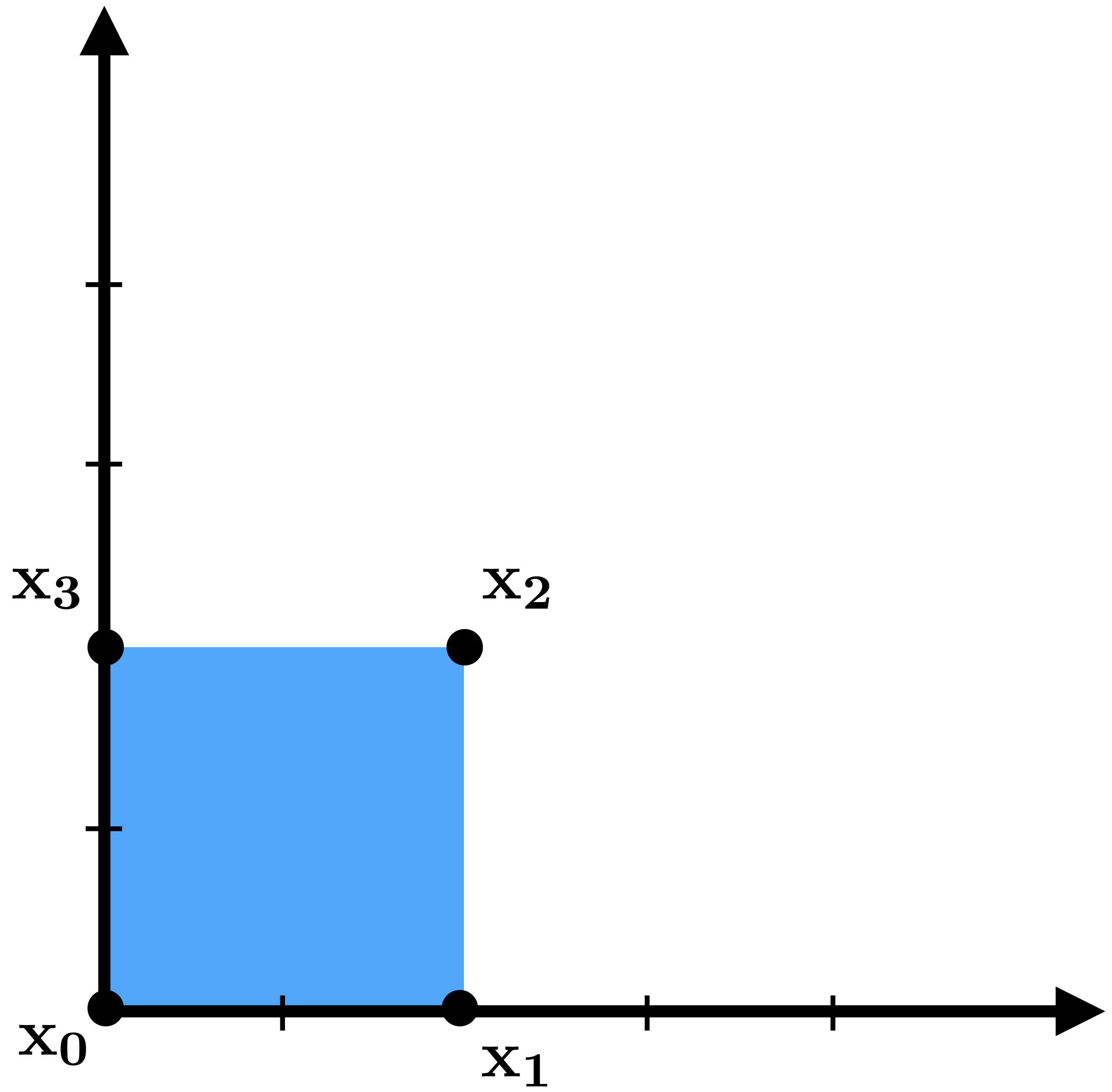
Linear transforms/maps—visualized

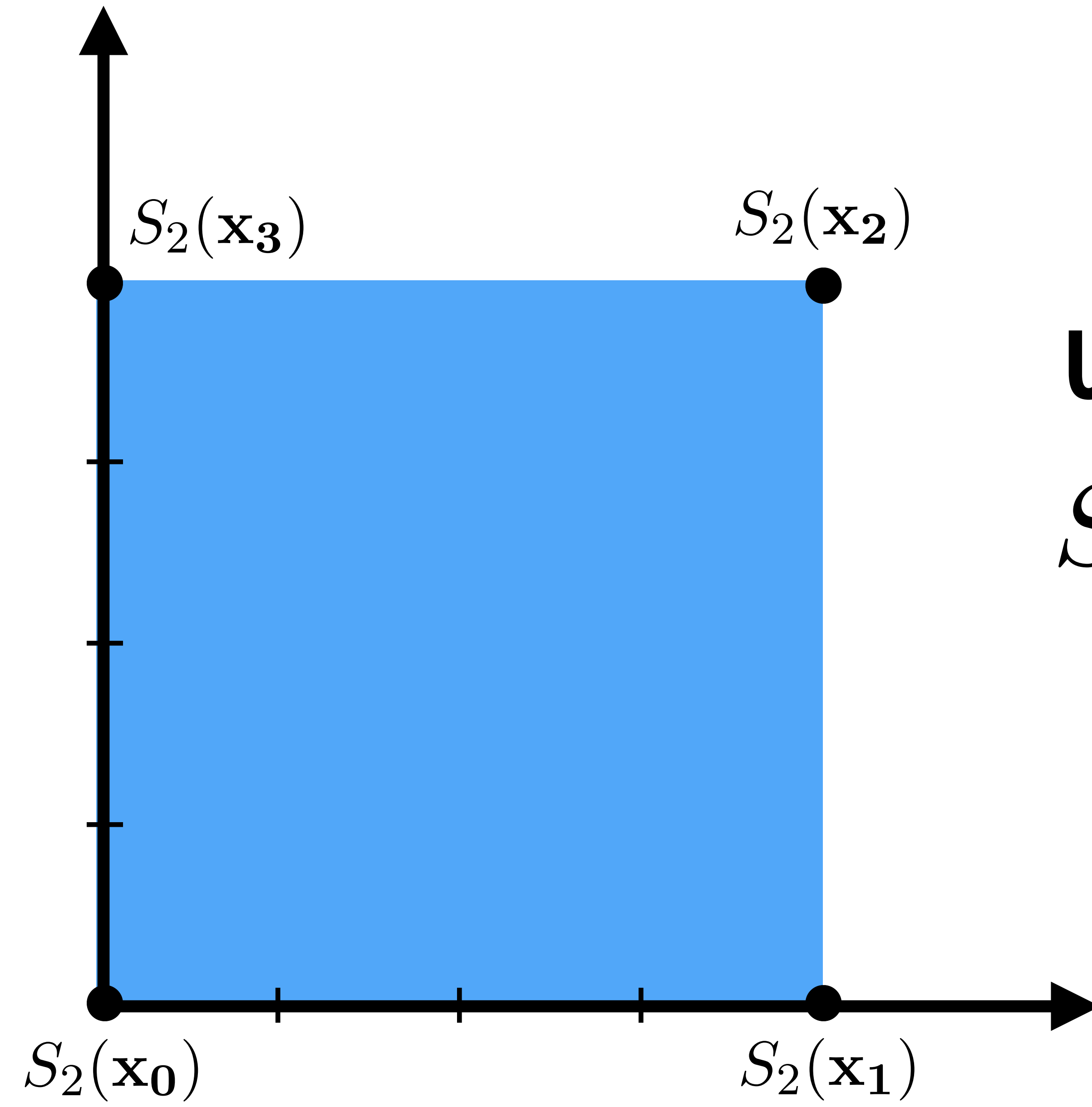
- **Example:**



Key idea: *linear maps take lines to lines*

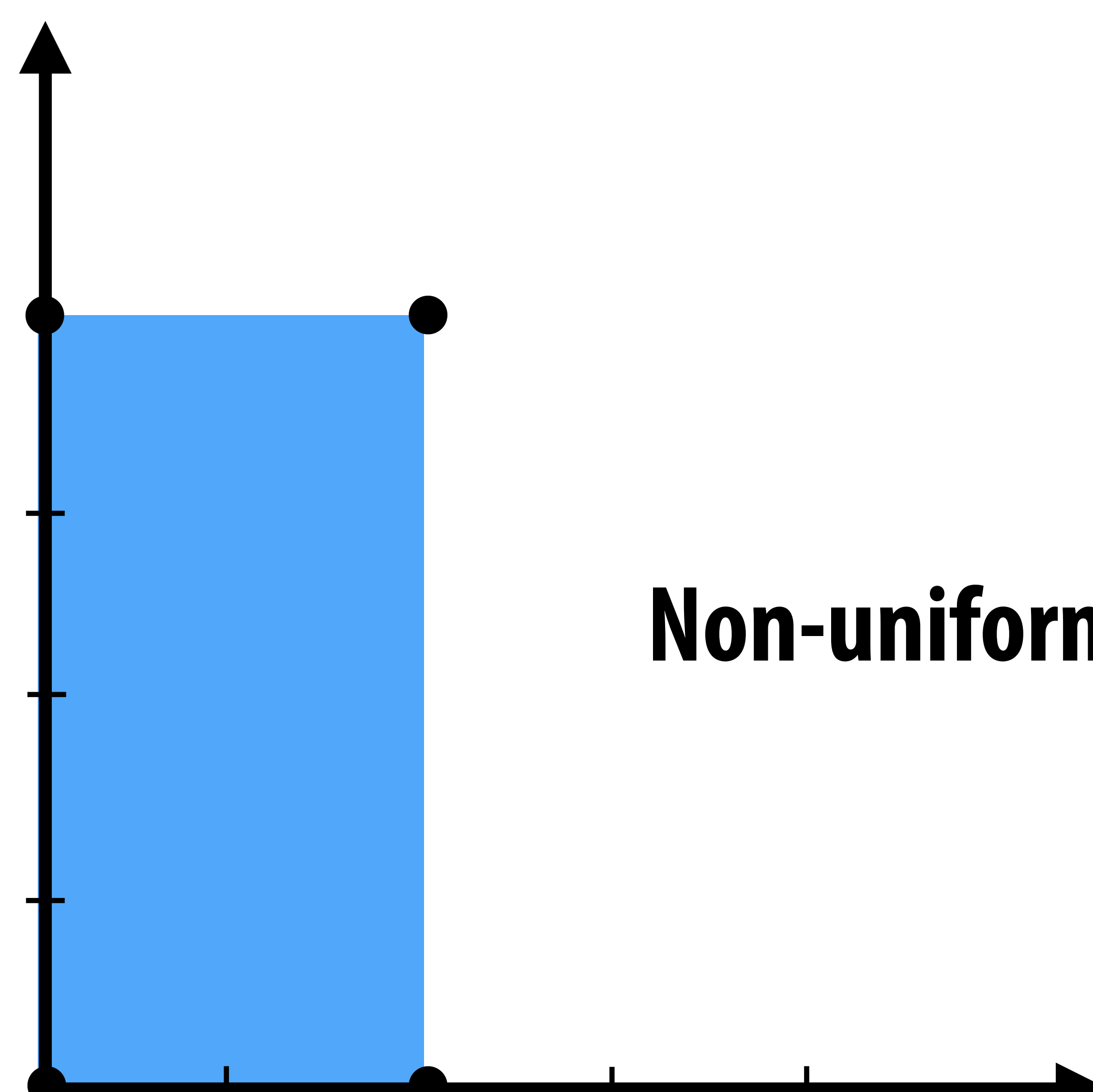
Scale





Uniform scale:

$$S_a(\mathbf{x}) = a\mathbf{x}$$



Non-uniform scale??

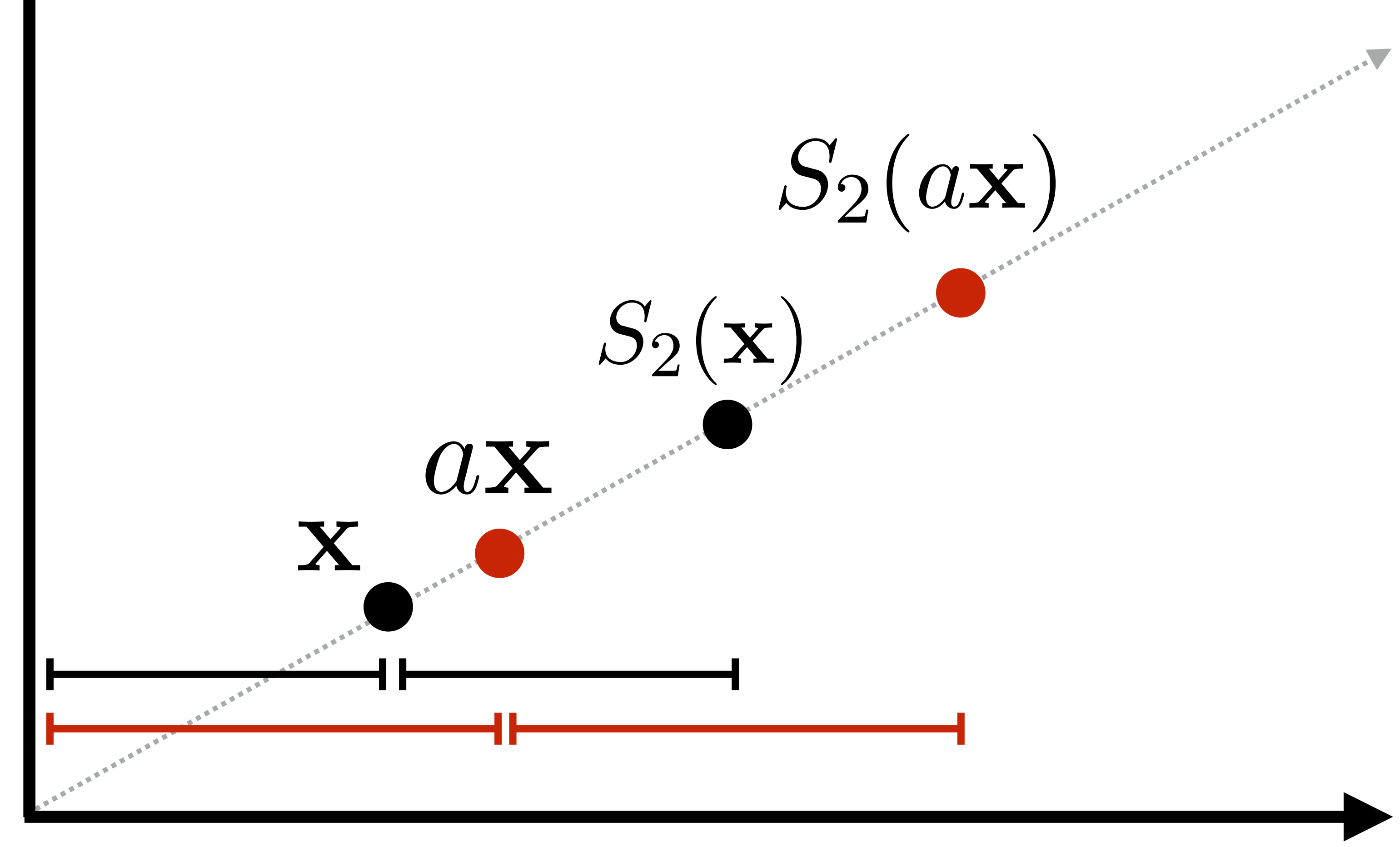
Is scale a linear transform?

$$S_2(\mathbf{x}) = 2\mathbf{x}$$

$$aS_2(\mathbf{x}) = 2a\mathbf{x}$$

$$S_2(a\mathbf{x}) = 2a\mathbf{x}$$

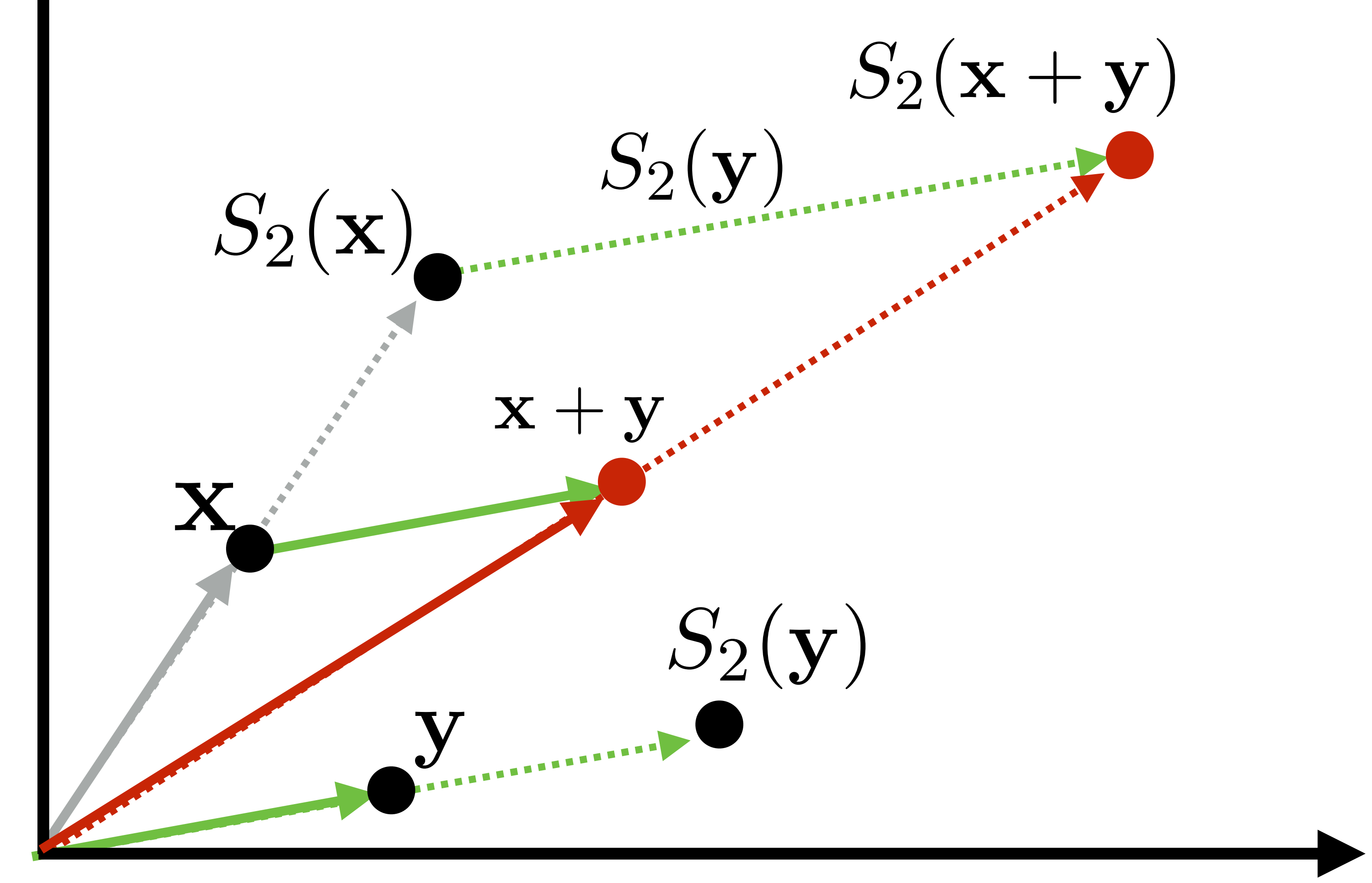
$$S_2(a\mathbf{x}) = aS_2(\mathbf{x})$$



$$S_2(\mathbf{x}+\mathbf{y}) = 2(\mathbf{x}+\mathbf{y})$$

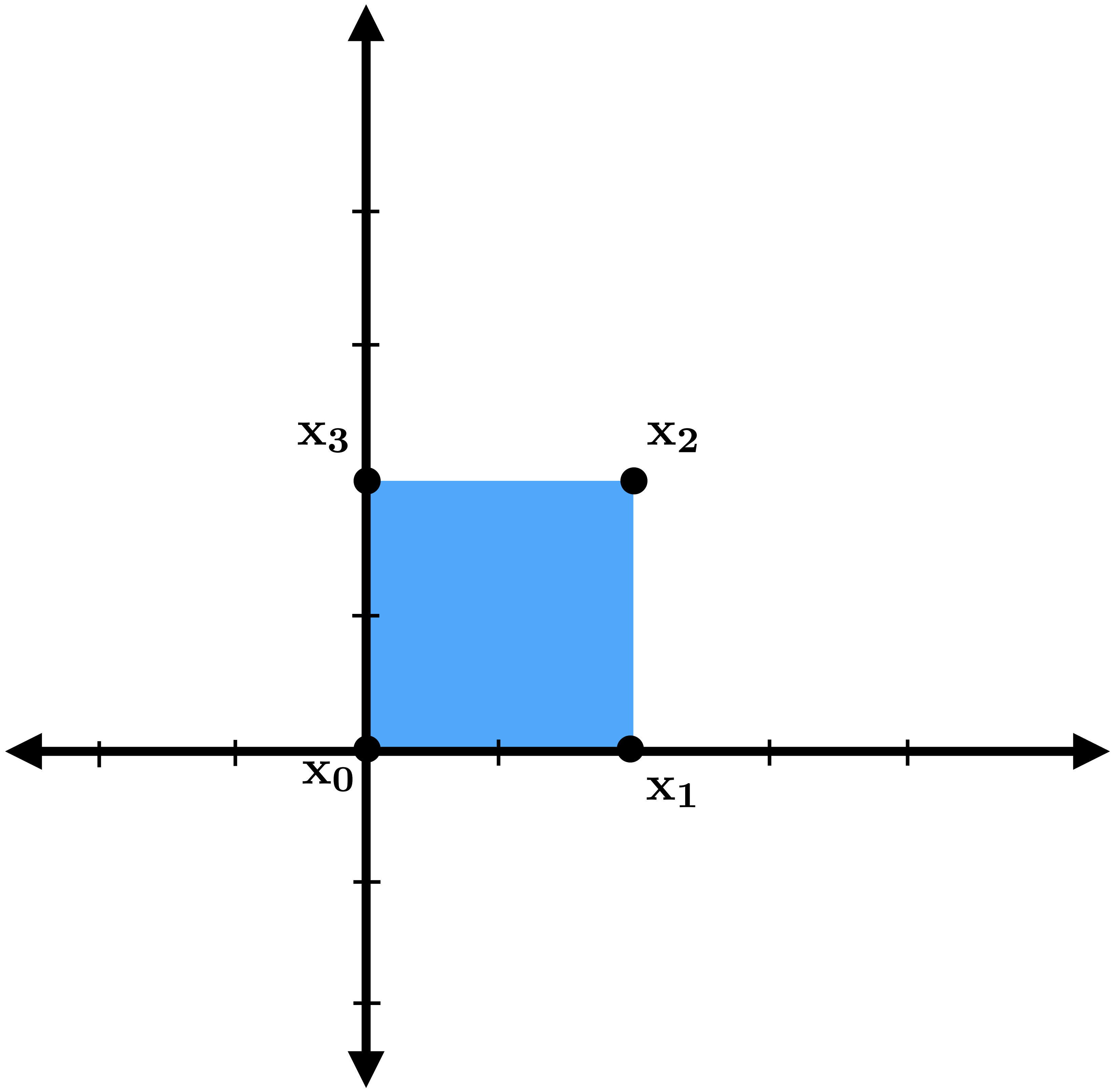
$$S_2(\mathbf{x}) + S_2(\mathbf{y}) = 2\mathbf{x} + 2\mathbf{y}$$

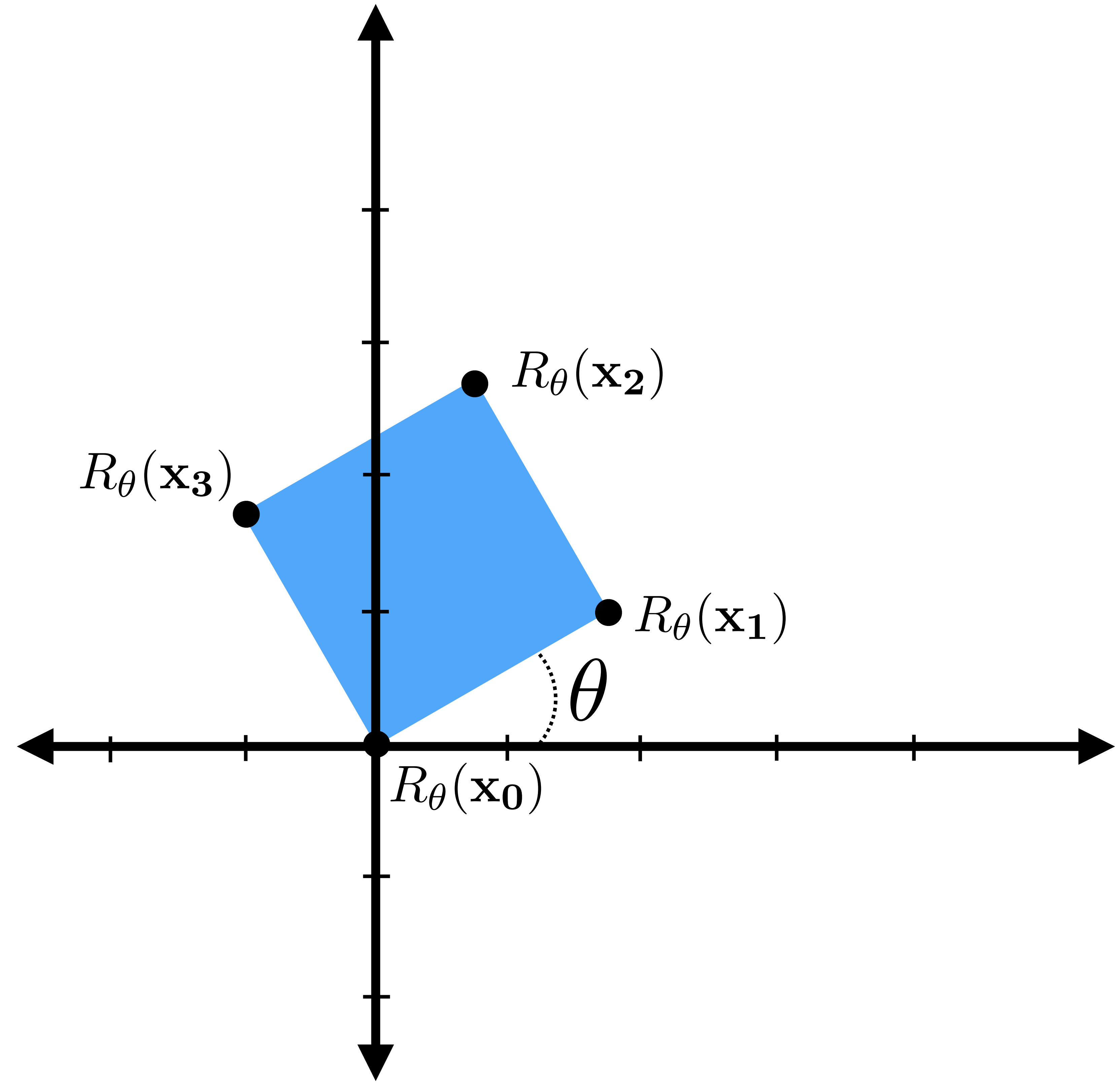
$$S_2(\mathbf{x}+\mathbf{y}) = S_2(\mathbf{x}) + S_2(\mathbf{y})$$



Yes!

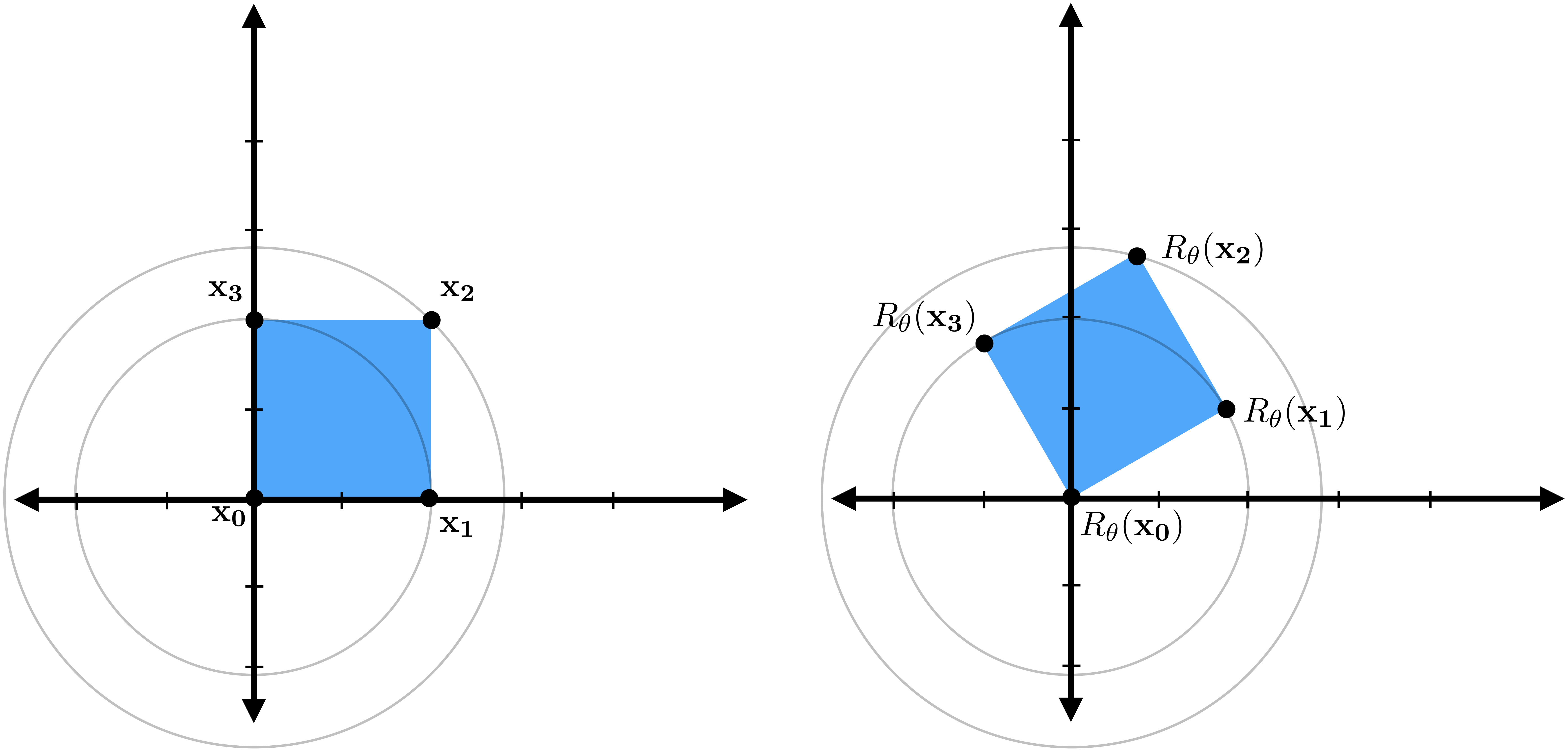
Rotation





R_θ **= rotate counter-clockwise by** θ

Rotation as circular motion

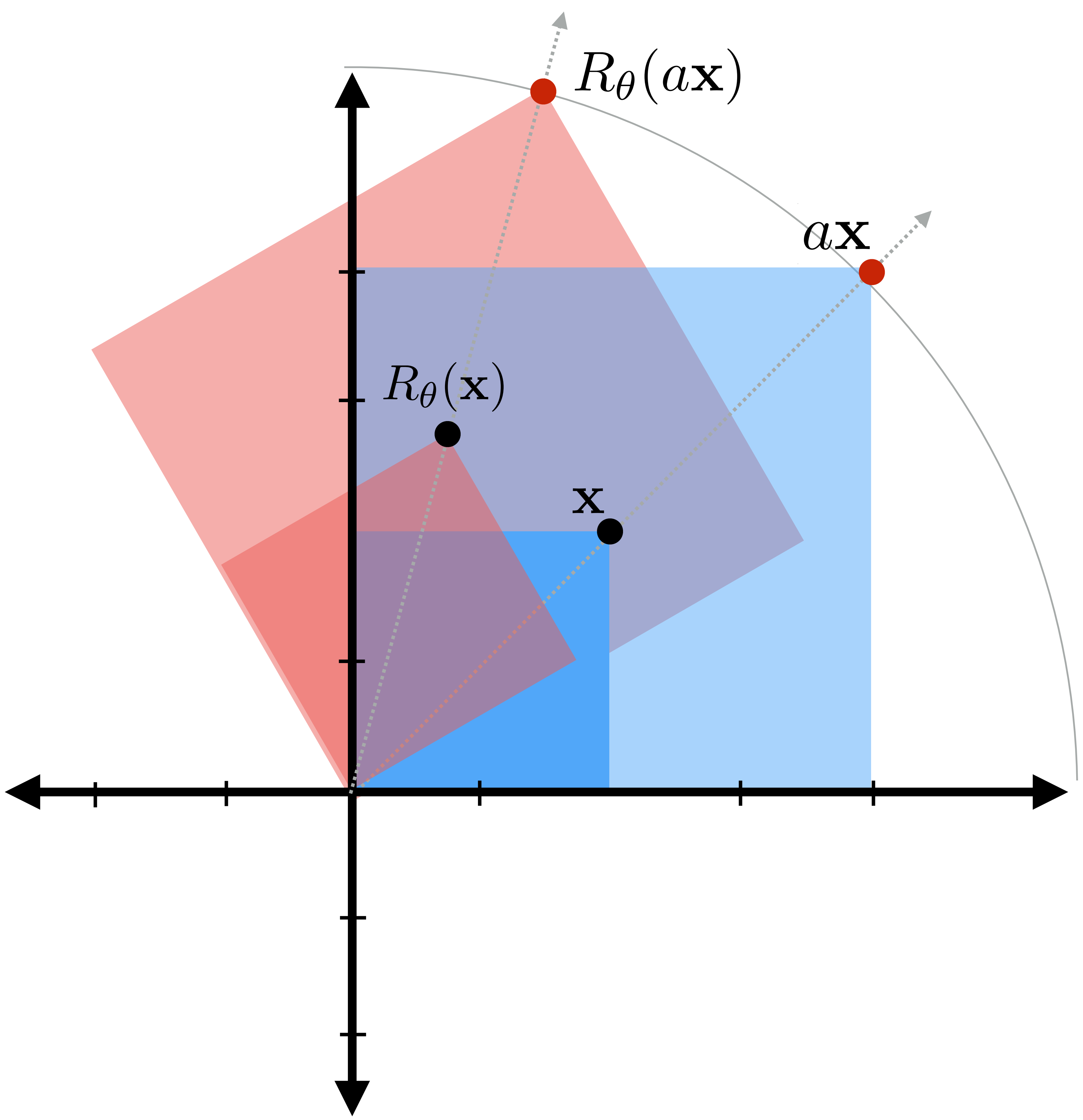


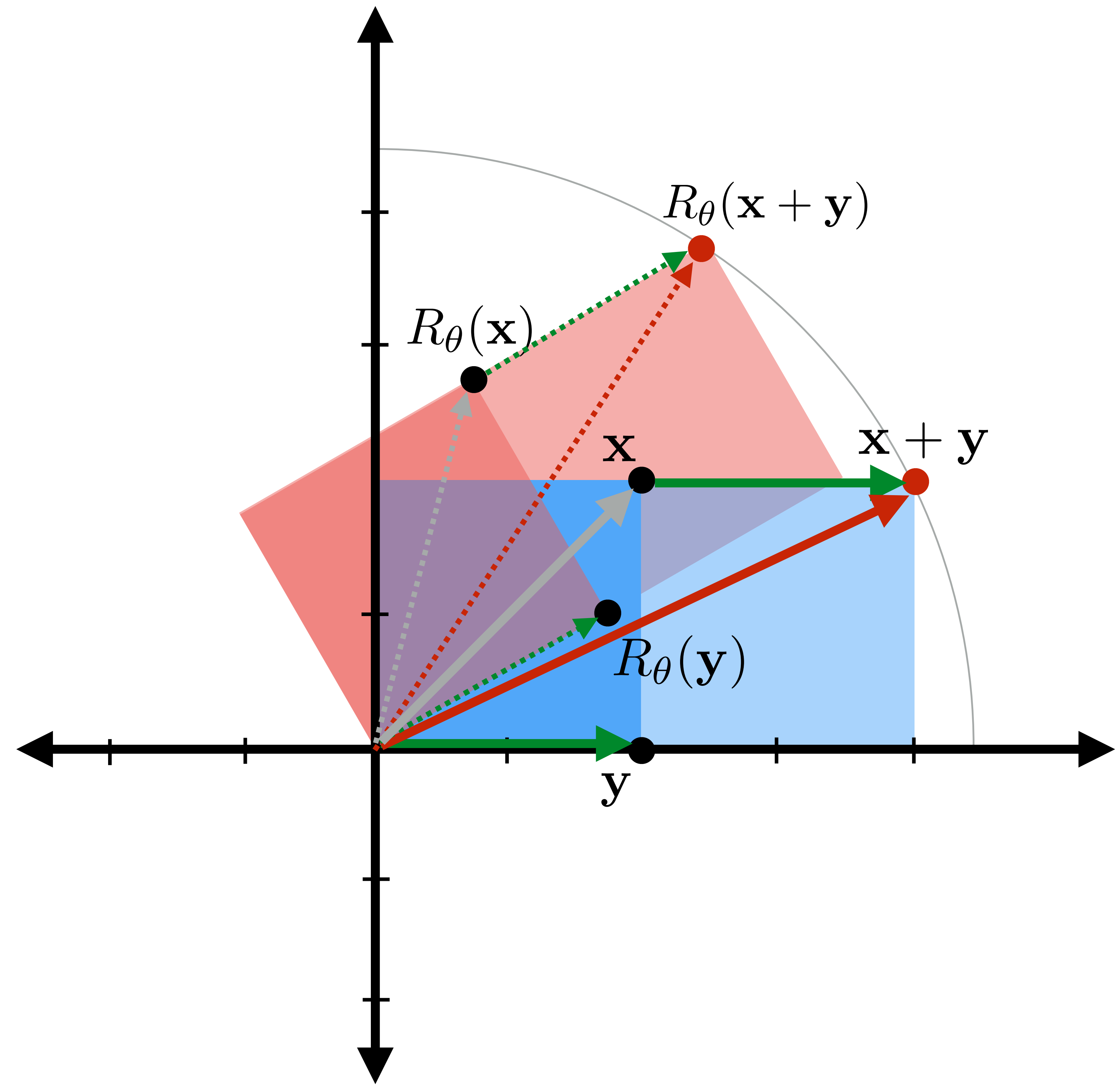
R_θ = rotate counter-clockwise by θ

As angle changes, points move along *circular* trajectories.

Hence, rotations preserve length of vectors: $|R_\theta(\mathbf{x})| = |\mathbf{x}|$

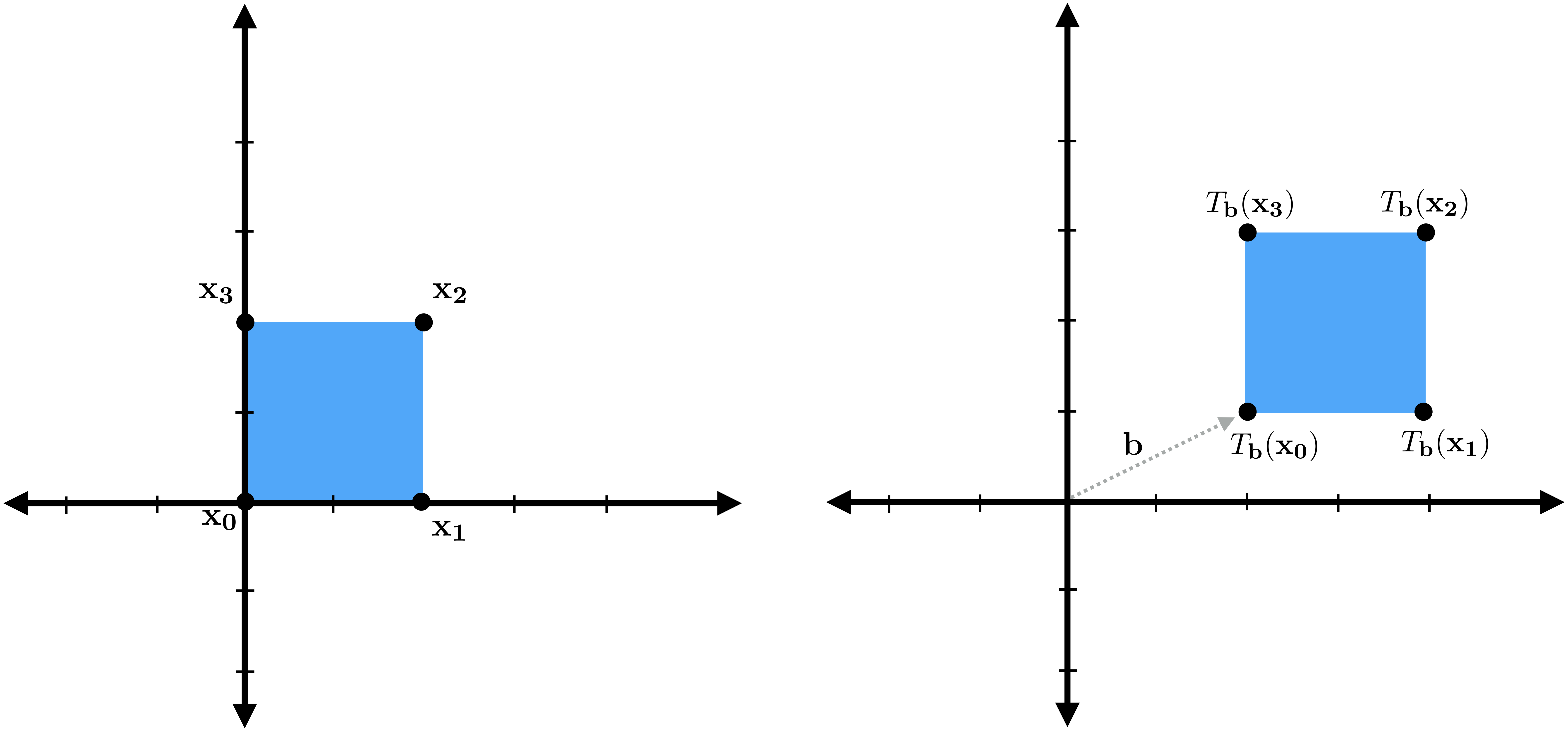
Is rotation linear?





Yes!

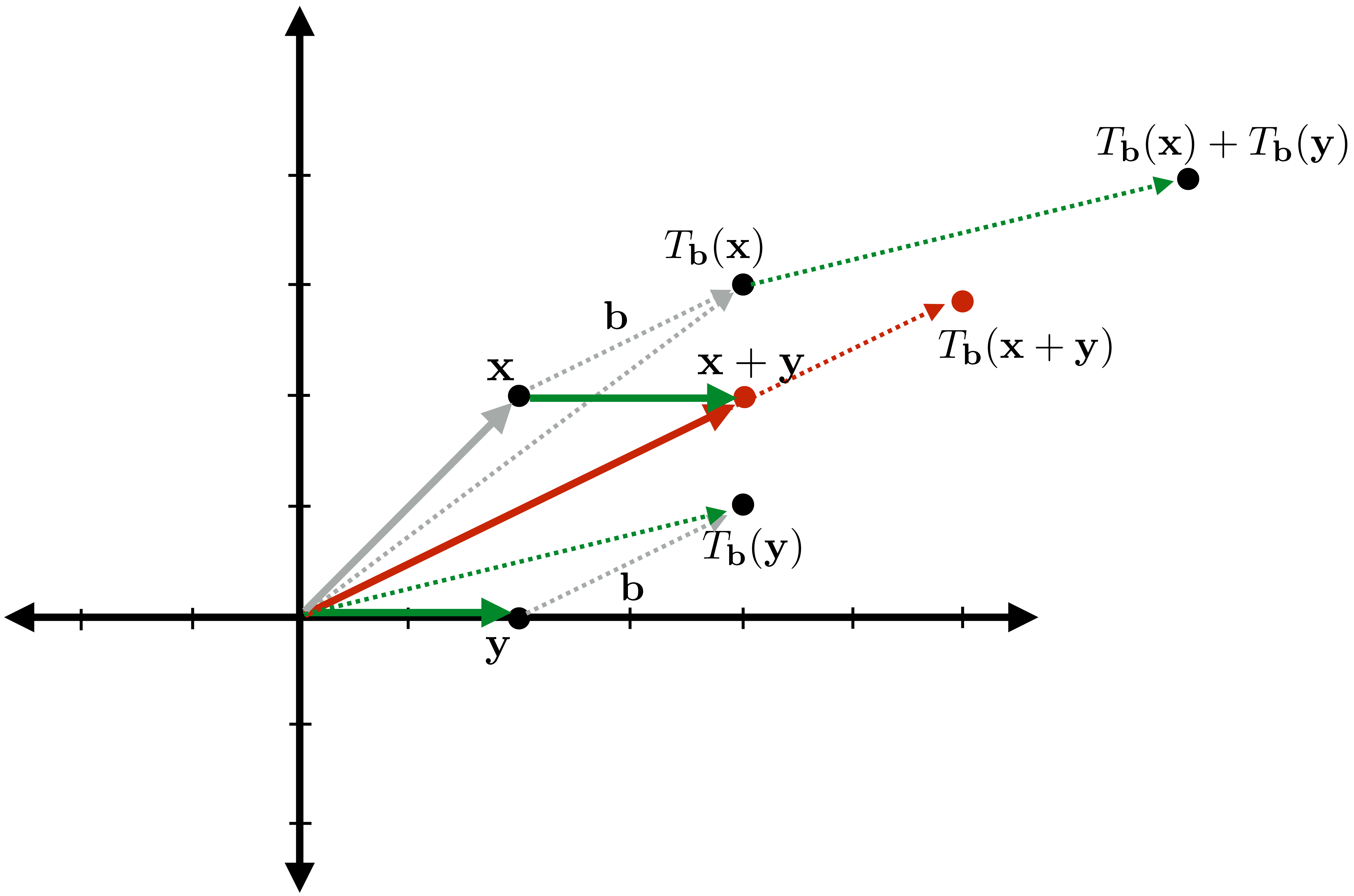
Translation



$T_{\mathbf{b}}$ — **"translate by b"**

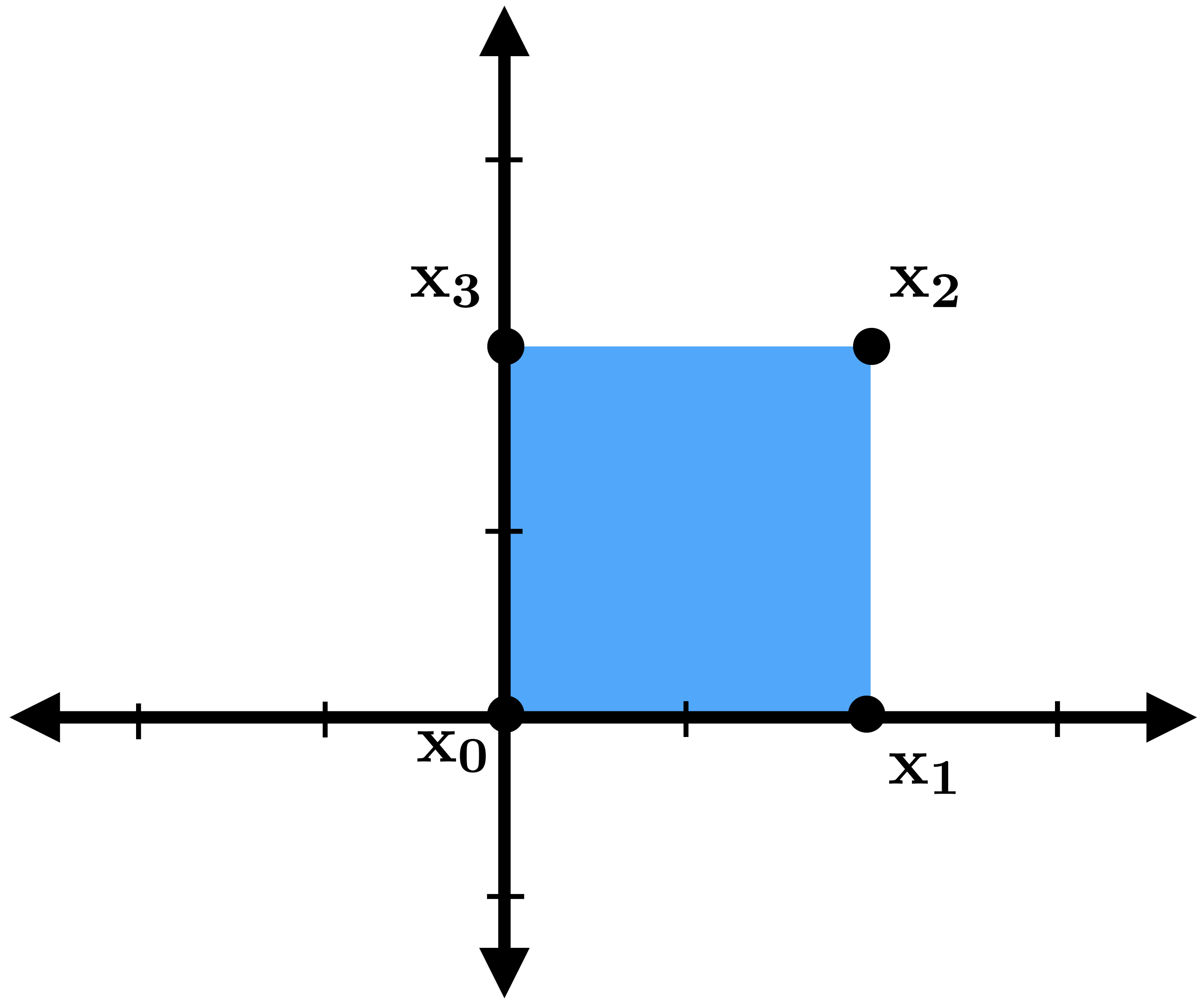
$$T_{\mathbf{b}}(\mathbf{x}) = \mathbf{x} + \mathbf{b}$$

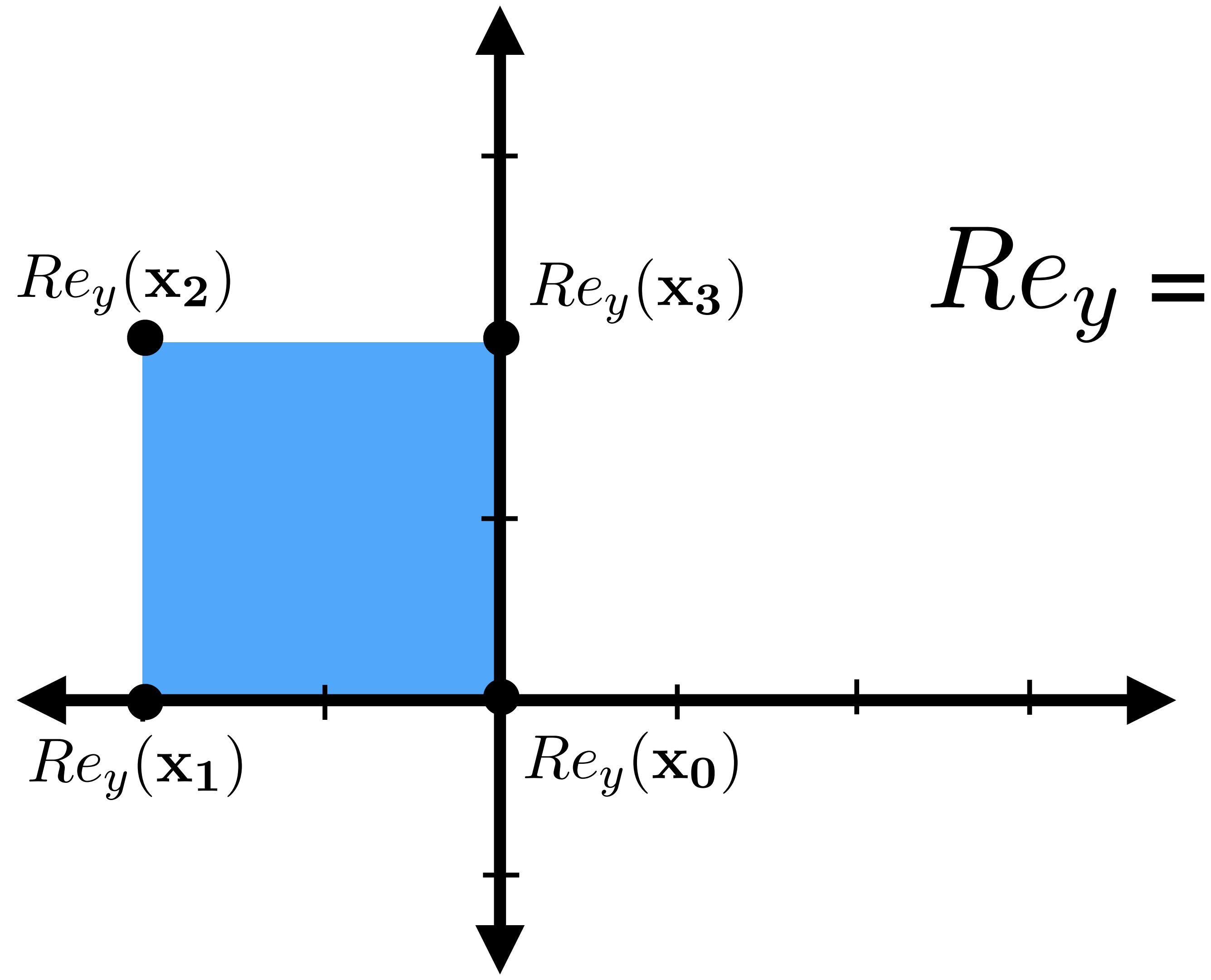
Is translation linear?



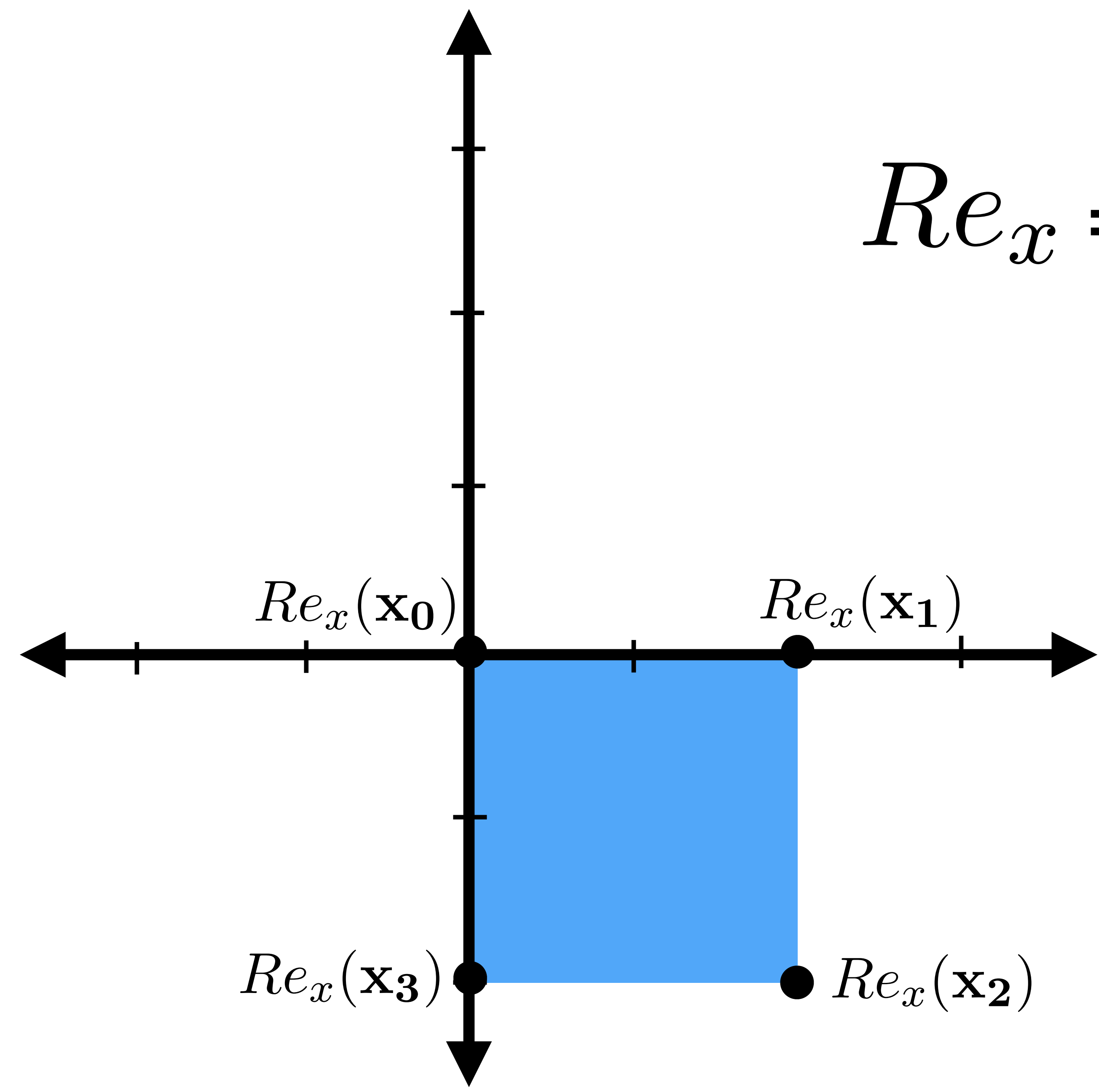
No. Translation is affine.

Reflection



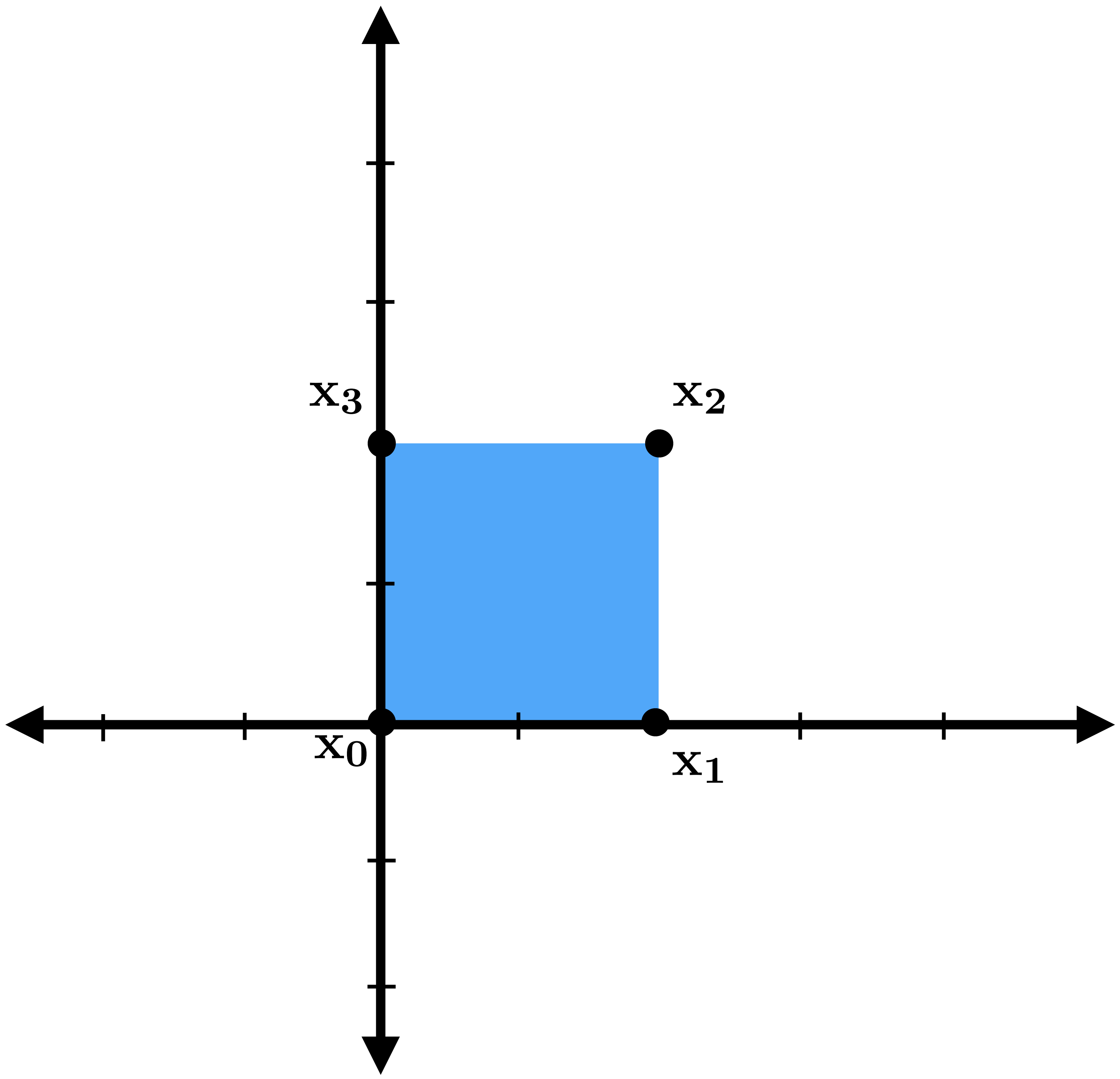


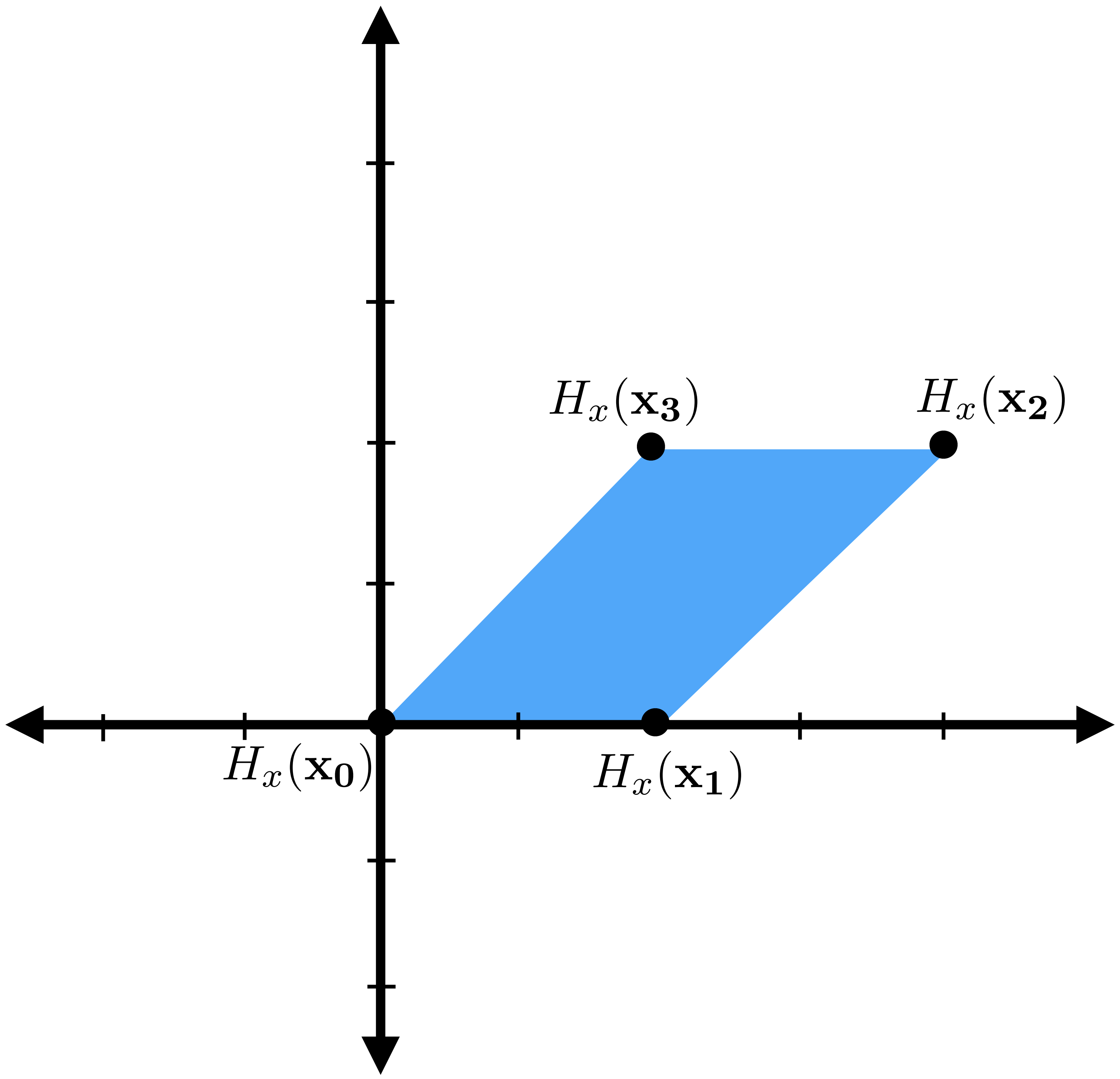
Re_y = **reflection about *y***



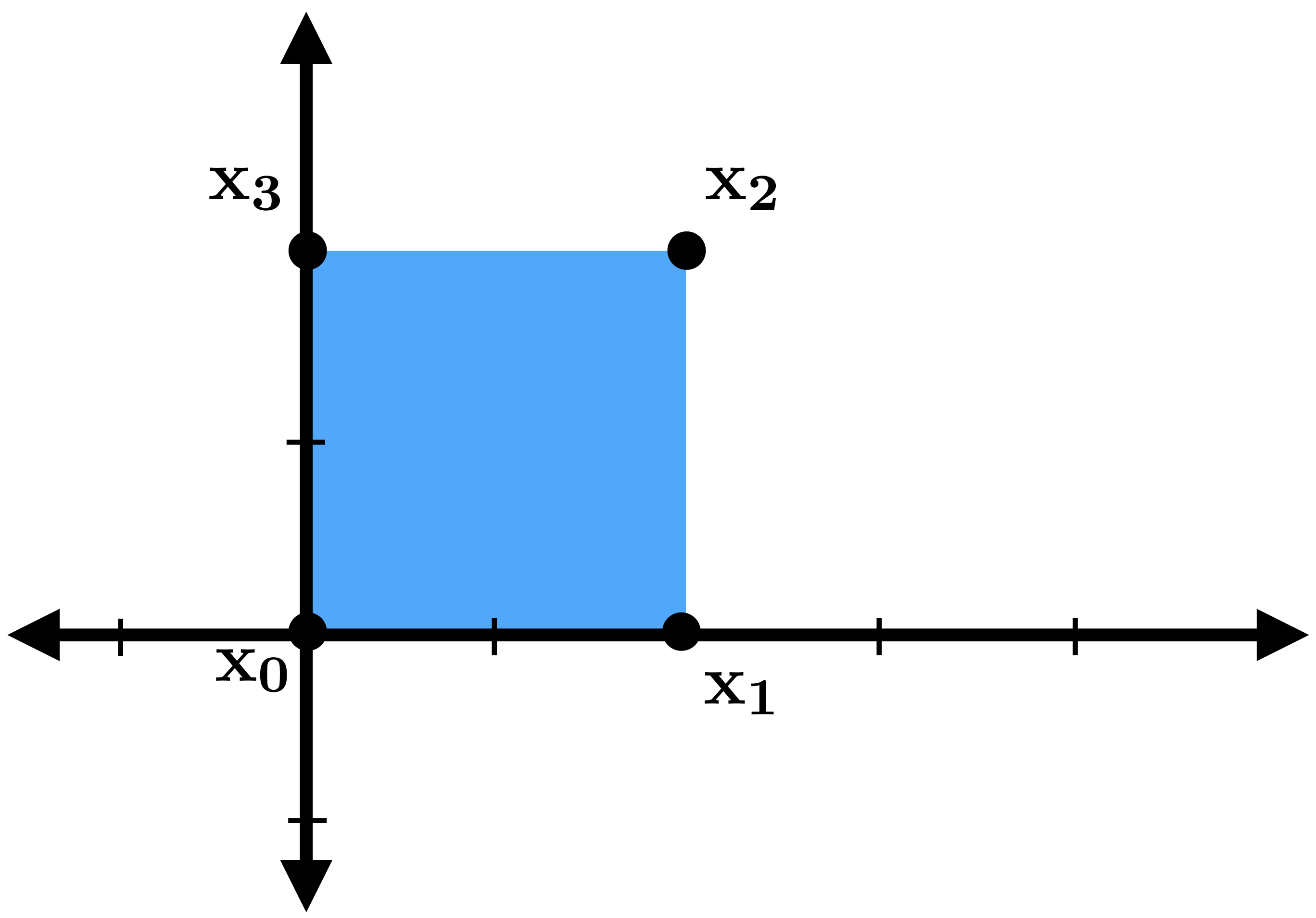
Re_x = **reflection about *x***

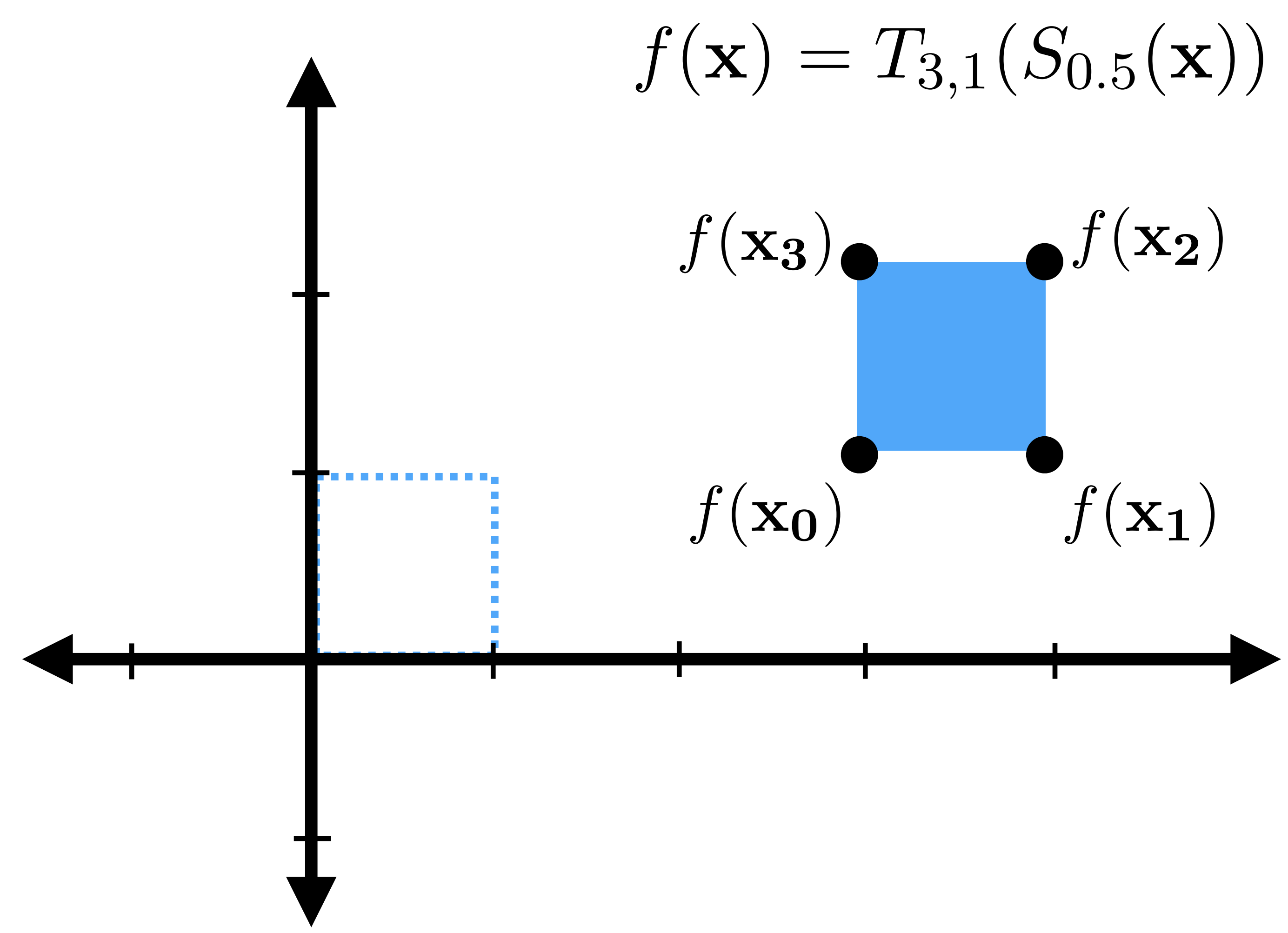
Shear (in x direction)

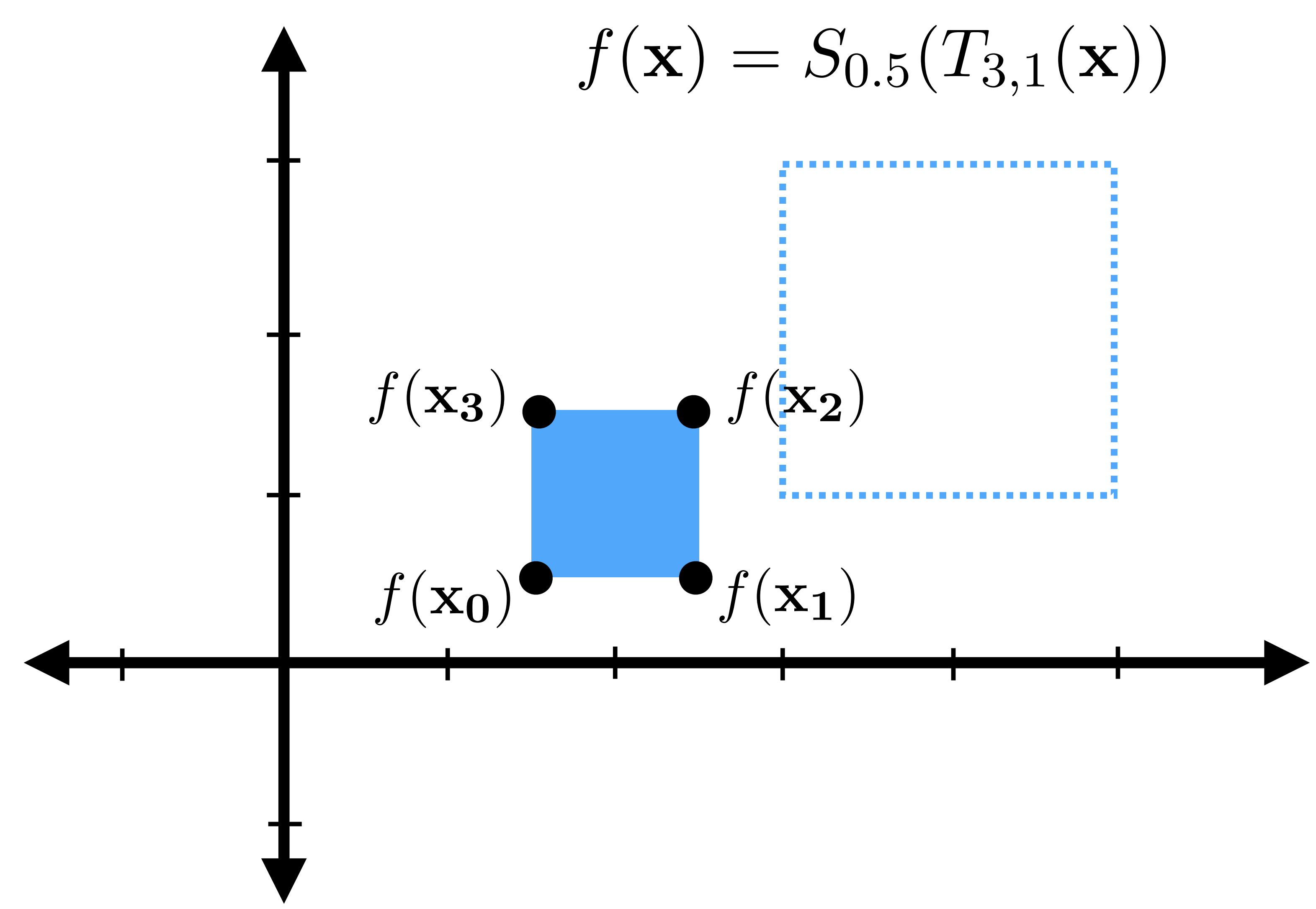




Compose basic transformations to construct more complicated ones



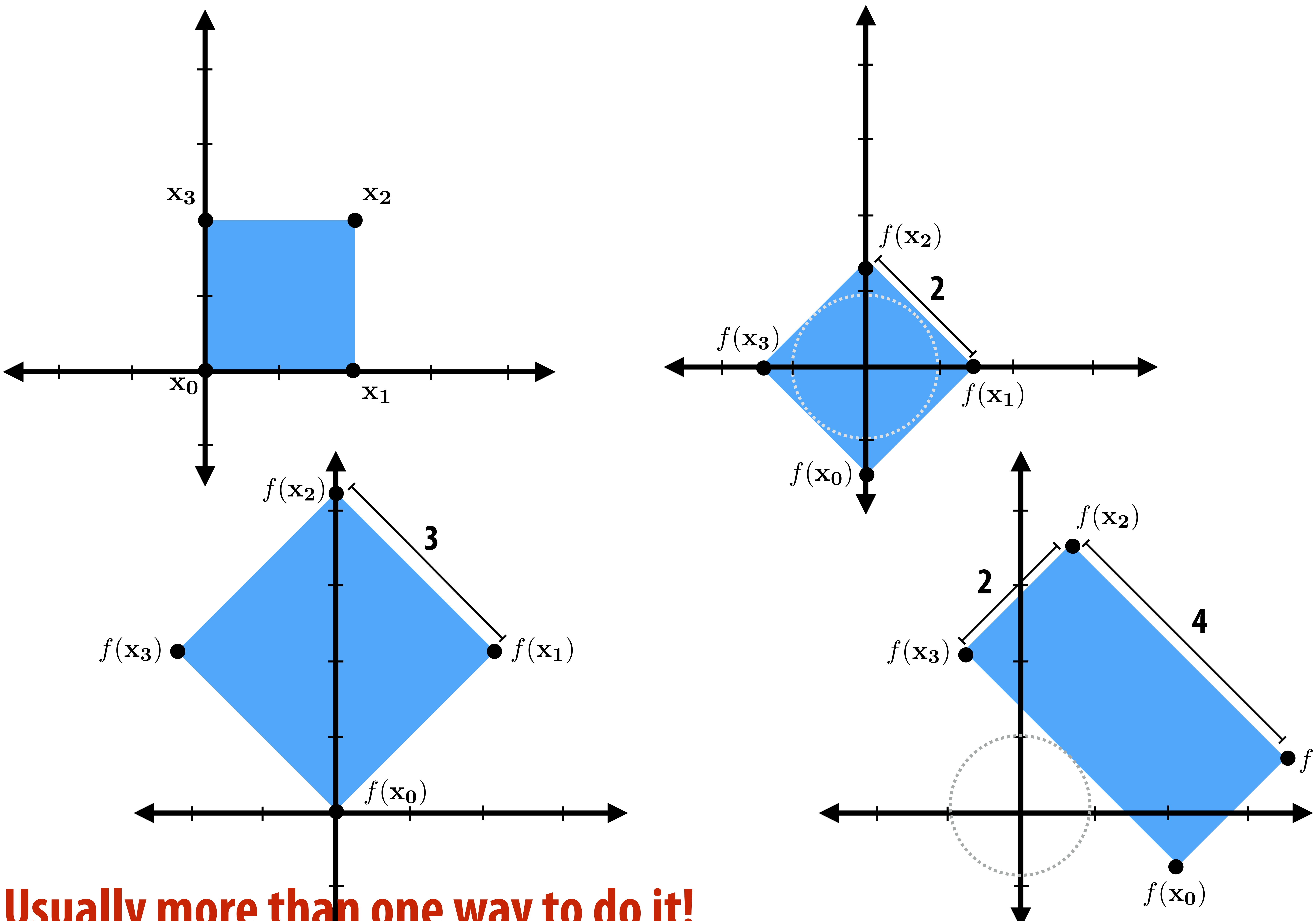




Note: order of composition matters

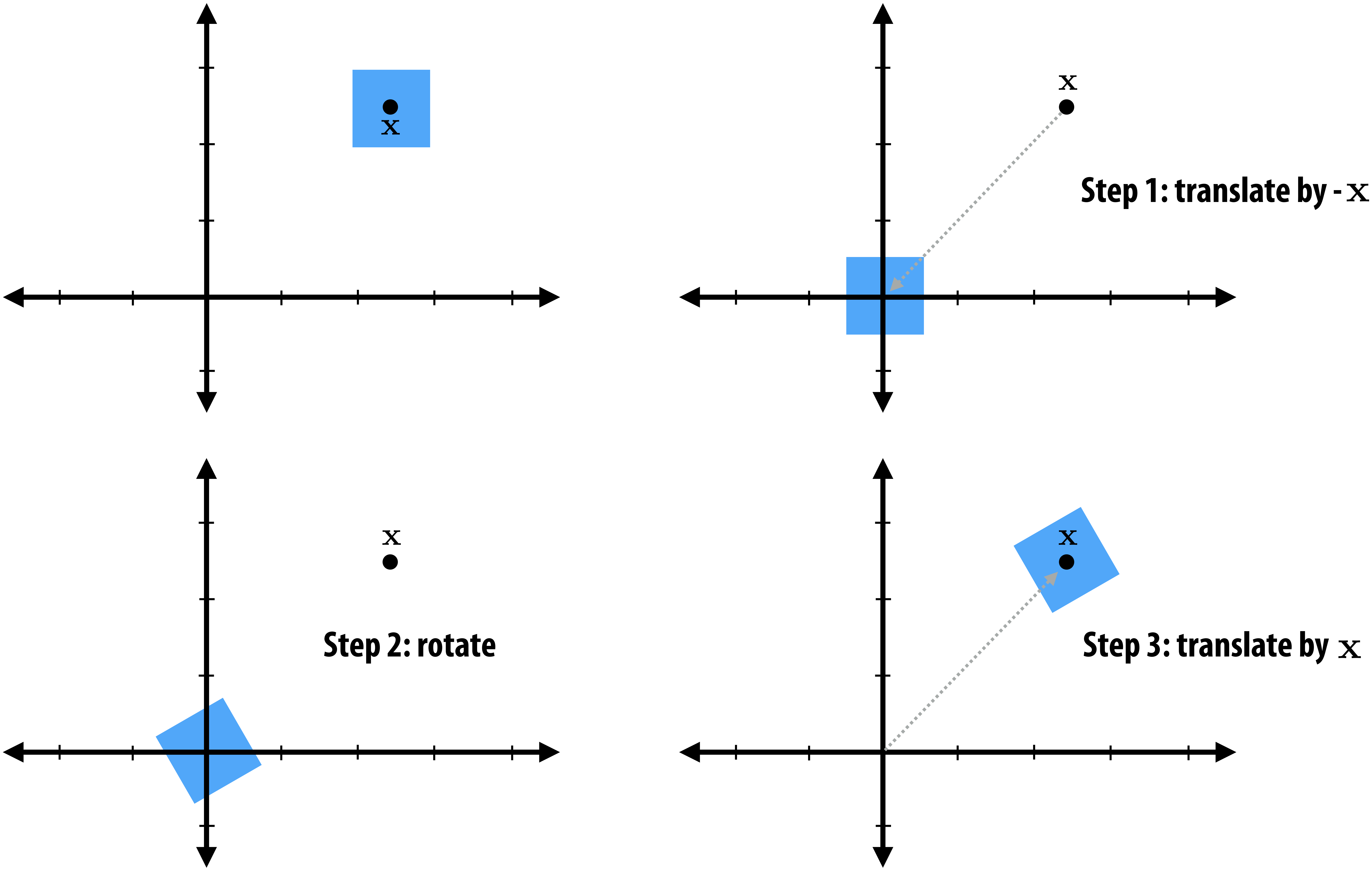
Top-right: scale, then translate
Bottom-right: translate, then scale

How would you perform these transformations?



Usually more than one way to do it!

Common task: rotate about a point x



Summary of basic transformations

Linear:

$$f(\mathbf{x}+\mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$$
$$f(a\mathbf{x}) = af(\mathbf{x})$$

Scale
Rotation
Reflection
Shear

Not linear:

Translation

Affine:

Composition of linear transform + translation
(all examples on previous two slides)

$$f(\mathbf{x}) = g(\mathbf{x}) + \mathbf{b}$$

Not affine: perspective projection (will discuss later)

Euclidean: (Isometries)

Preserve distance between points (preserves length)

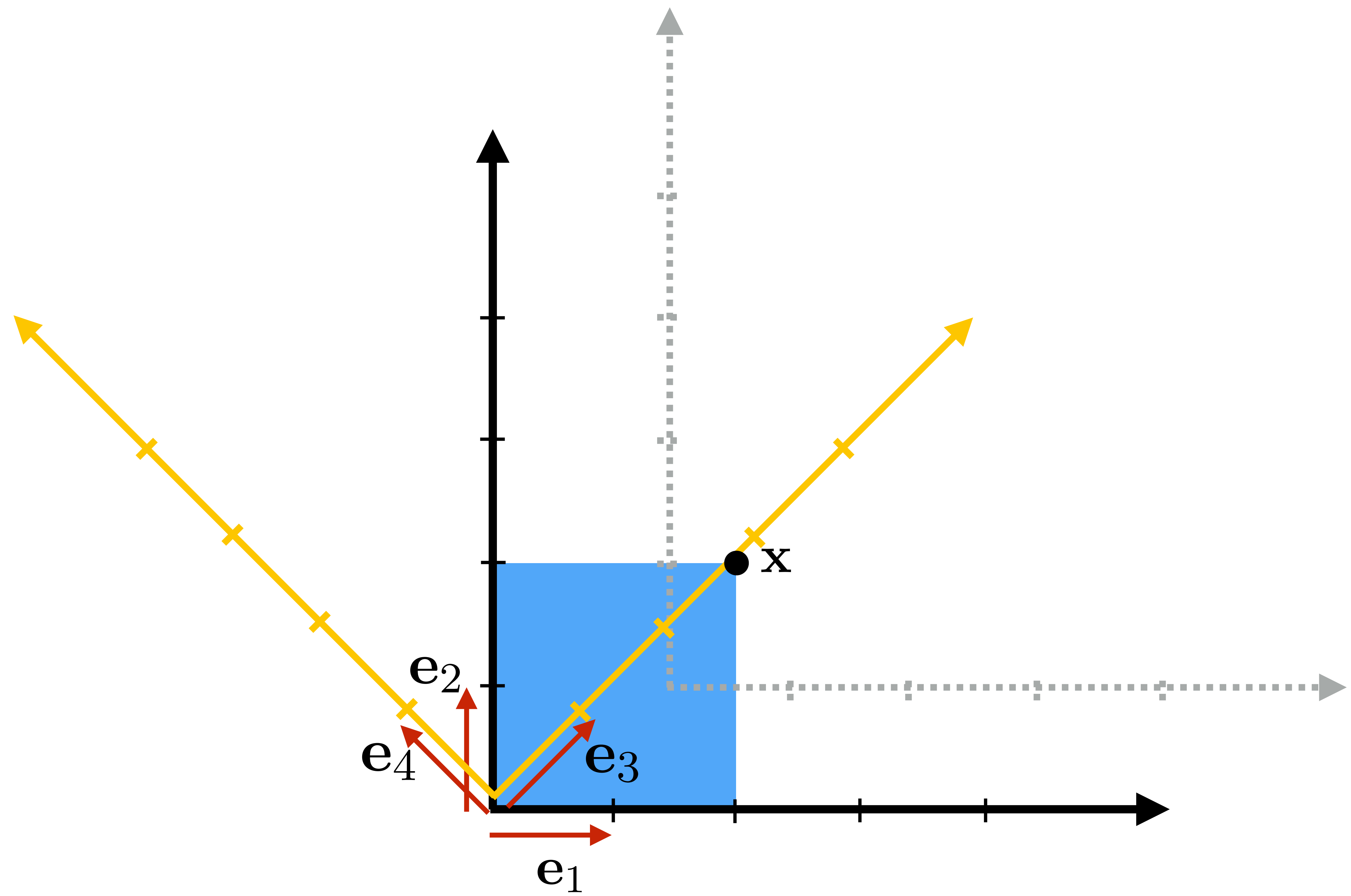
$$|f(\mathbf{x}) - f(\mathbf{y})| = |\mathbf{x} - \mathbf{y}|$$

Translation
Rotation
Reflection

"Rigid body" transformations are distance-preserving motions that also preserve *orientation* (i.e., does not include reflection)

Representing Transformations in Coordinates

Review: representing points in a coordinate space



Consider coordinate space defined by orthogonal vectors $\mathbf{e}_1$ and $\mathbf{e}_2$

$$\mathbf{x} = 2\mathbf{e}_1 + 2\mathbf{e}_2$$

$$\mathbf{x} = \begin{bmatrix} 2 & 2 \end{bmatrix}$$

$\mathbf{x} = \begin{bmatrix} 0.5 & 1 \end{bmatrix}$ in coordinate space defined by $\mathbf{e}_1$ and $\mathbf{e}_2$, with origin at (1.5, 1)

$\mathbf{x} = \begin{bmatrix} \sqrt{8} & 0 \end{bmatrix}$ in coordinate space defined by $\mathbf{e}_3$ and $\mathbf{e}_4$, with origin at (0, 0)

Review: 2D matrix multiplication

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} =$$

$$x \begin{bmatrix} a \\ c \end{bmatrix} + y \begin{bmatrix} b \\ d \end{bmatrix} =$$

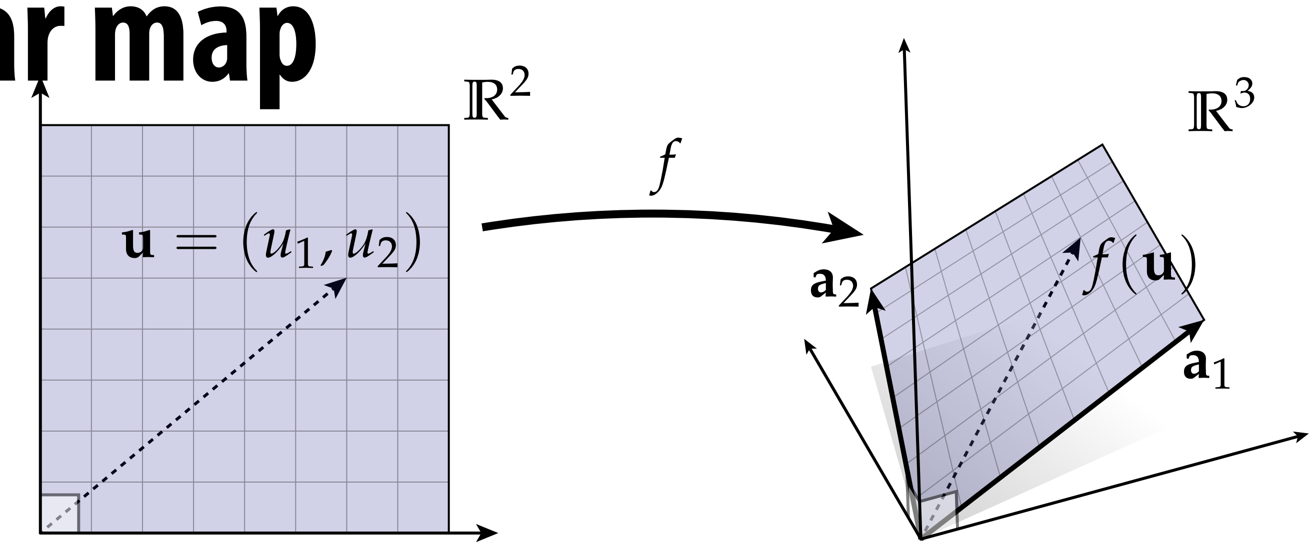
$$\begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

- **Matrix multiplication is linear combination of columns**
- **Encodes a linear map!**

Linear maps via matrices

- **Example: suppose I have a linear map**

$$f(\mathbf{u}) = u_1\mathbf{a}_1 + u_2\mathbf{a}_2$$



- **Encoding as a matrix: "a" vectors become matrix columns:**

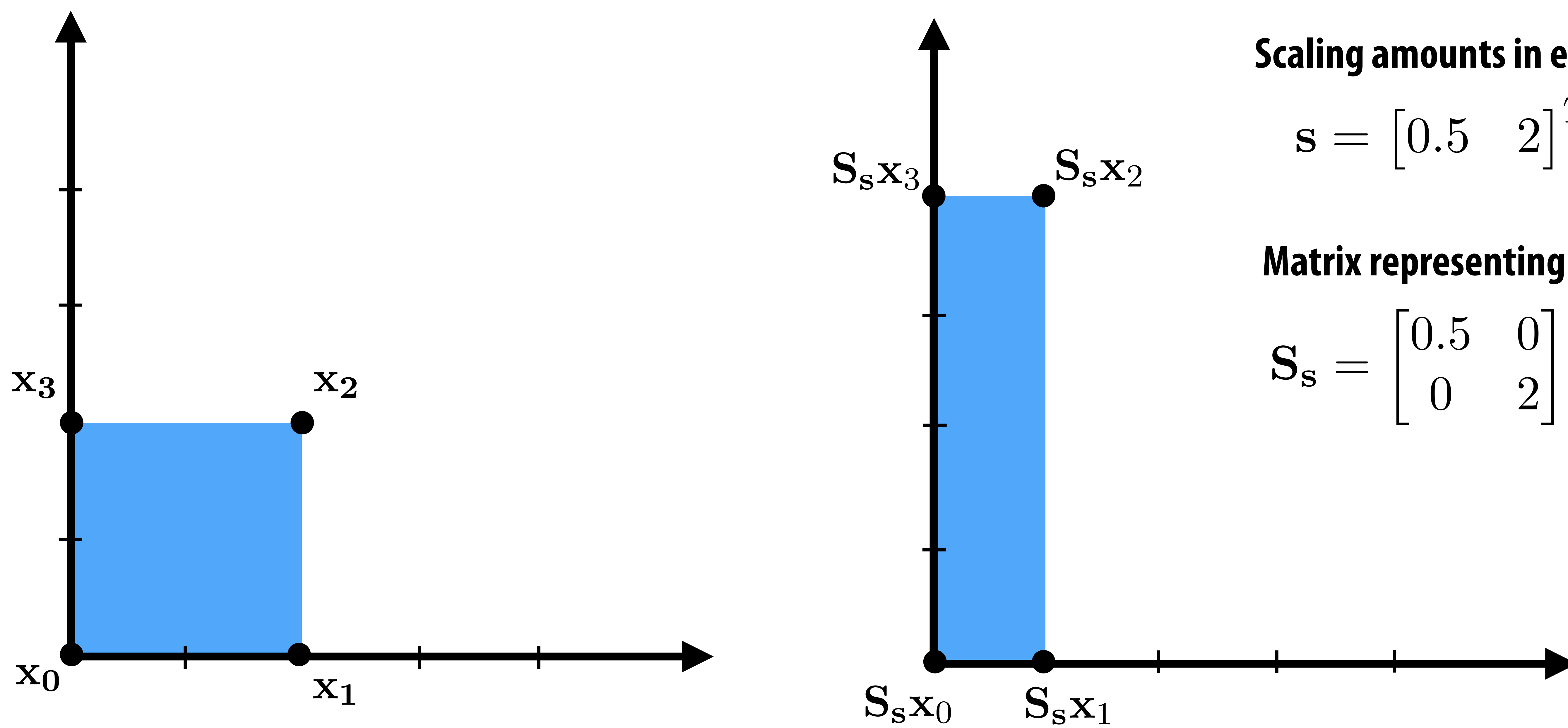
$$A := \begin{bmatrix} a_{1,x} & a_{2,x} \\ a_{1,y} & a_{2,y} \\ a_{1,z} & a_{2,z} \end{bmatrix}$$

- **Matrix-vector multiply computes same output as original map:**

$$\begin{bmatrix} a_{1,x} & a_{2,x} \\ a_{1,y} & a_{2,y} \\ a_{1,z} & a_{2,z} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} a_{1,x}u_1 + a_{2,x}u_2 \\ a_{1,y}u_1 + a_{2,y}u_2 \\ a_{1,z}u_1 + a_{2,x}u_2 \end{bmatrix} = u_1\mathbf{a}_1 + u_2\mathbf{a}_2$$

Linear transformations in 2D can be represented as 2x2 matrices

Consider non-uniform scale: $\mathrm{S}_{\mathbf{s}} = \begin{bmatrix} \mathbf{s}_x & 0 \\ 0 & \mathbf{s}_y \end{bmatrix}$



Scaling amounts in each direction:

$$\mathbf{s} = \begin{bmatrix} 0.5 & 2 \end{bmatrix}^T$$

Matrix representing scale transform:

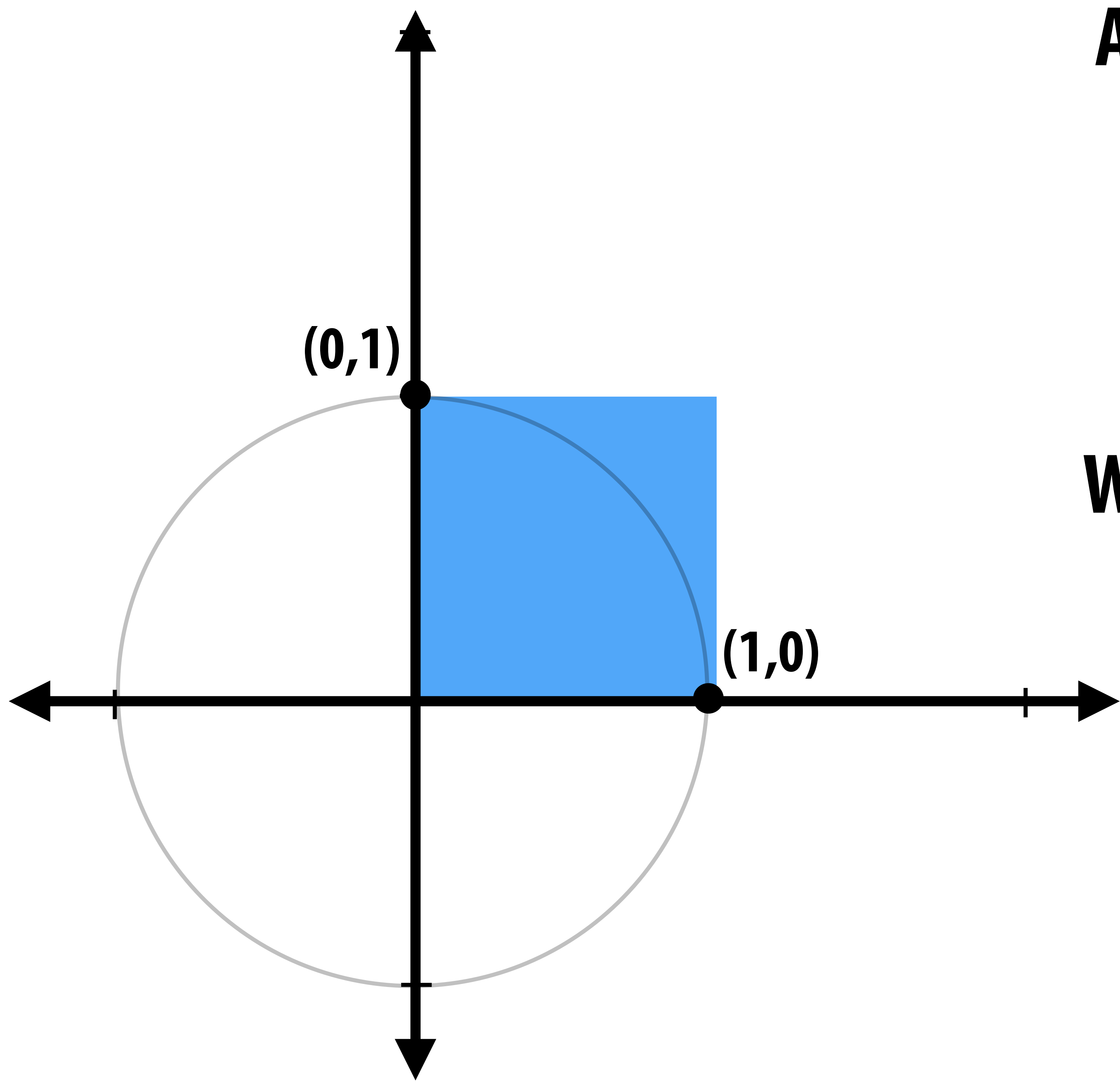
$$\mathbf{S_s} = \begin{bmatrix} 0.5 & 0 \\ 0 & 2 \end{bmatrix}$$

Rotation matrix (2D)

Question: what happens to (1, 0) and (0,1) after rotation by θ?

Reminder: rotation moves points along circular trajectories.

(Recall that $\cos\theta$ and $\sin\theta$ are the coordinates of a point on the unit circle.)



Answer:

$$R_\theta(1,0) = (\cos(\theta), \sin(\theta))$$

$$R_\theta(0,1) = (\cos(\theta + \pi/2), \sin(\theta + \pi/2))$$

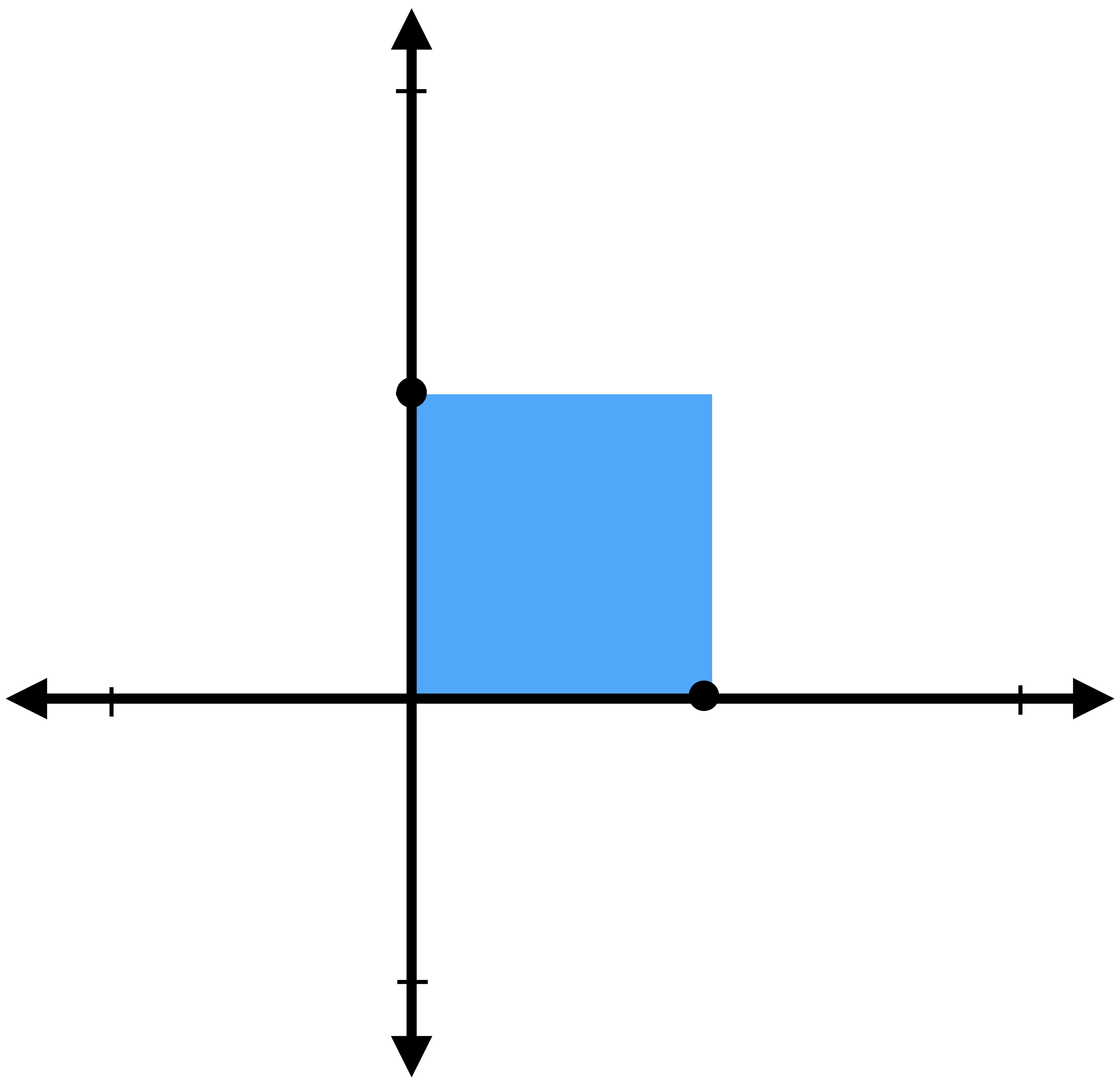
Which means the matrix must look like:

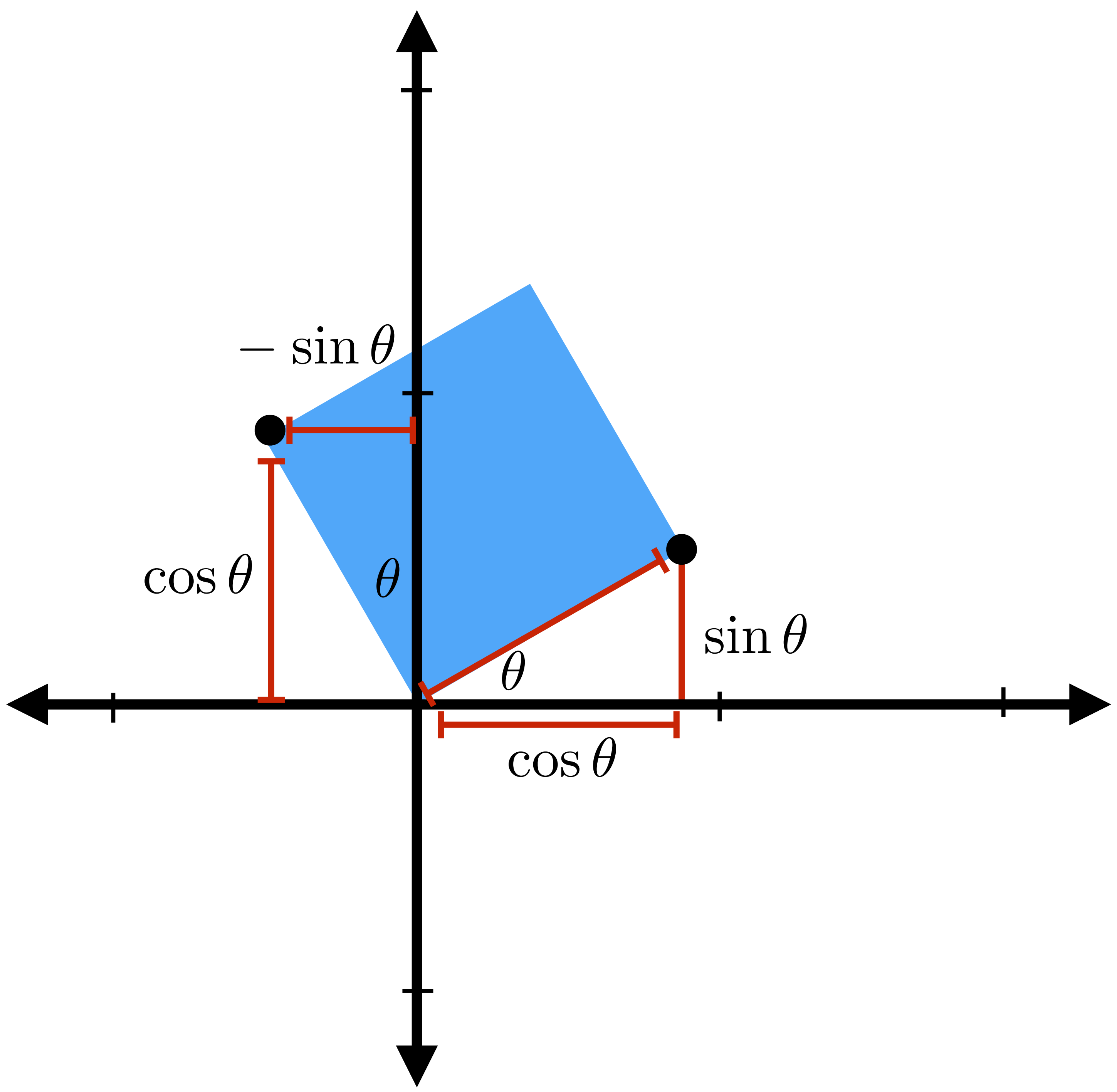
$$R_\theta = \begin{bmatrix} \cos(\theta) & \cos(\theta + \pi/2) \\ \sin(\theta) & \sin(\theta + \pi/2) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

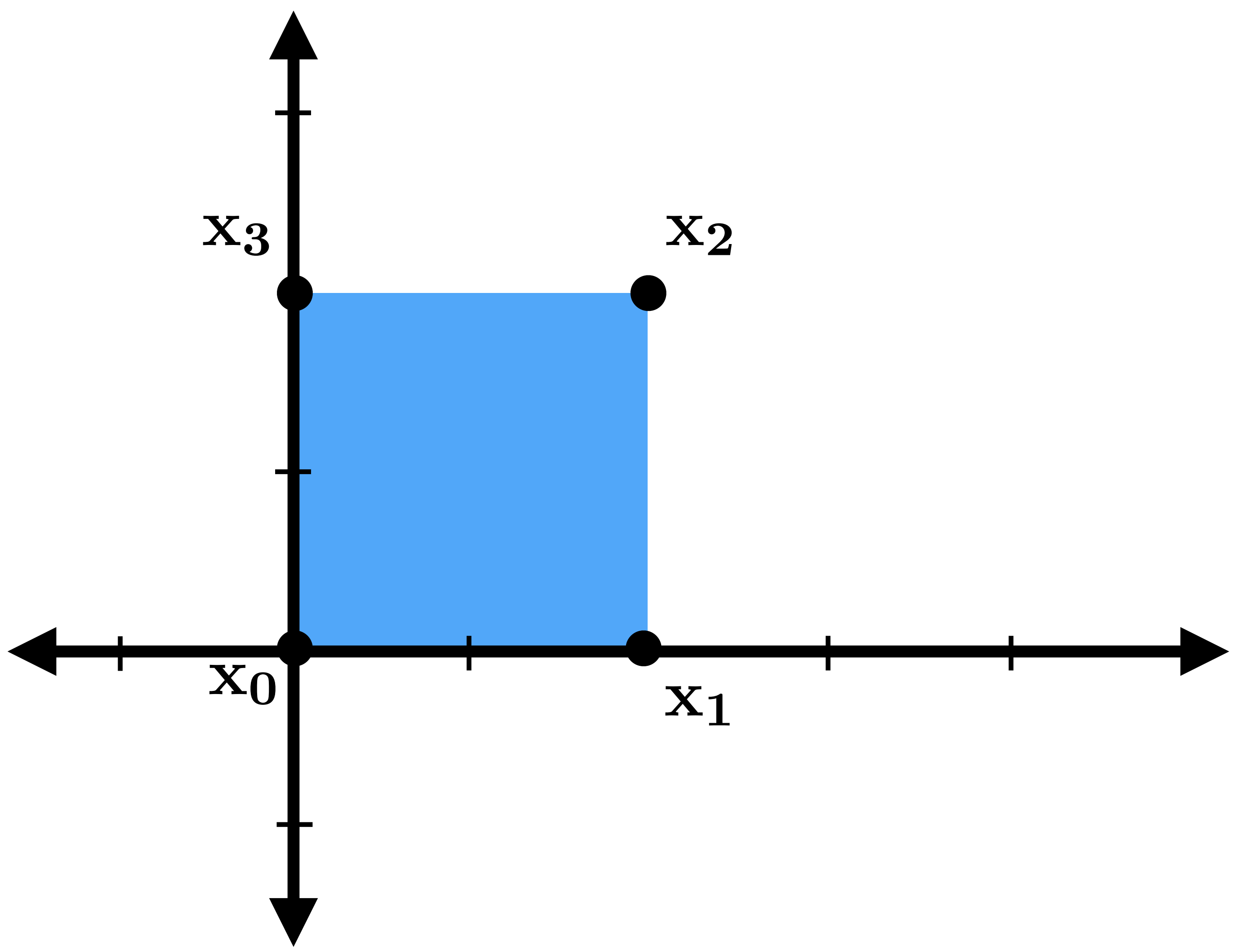
Rotation matrix (2D): another way…

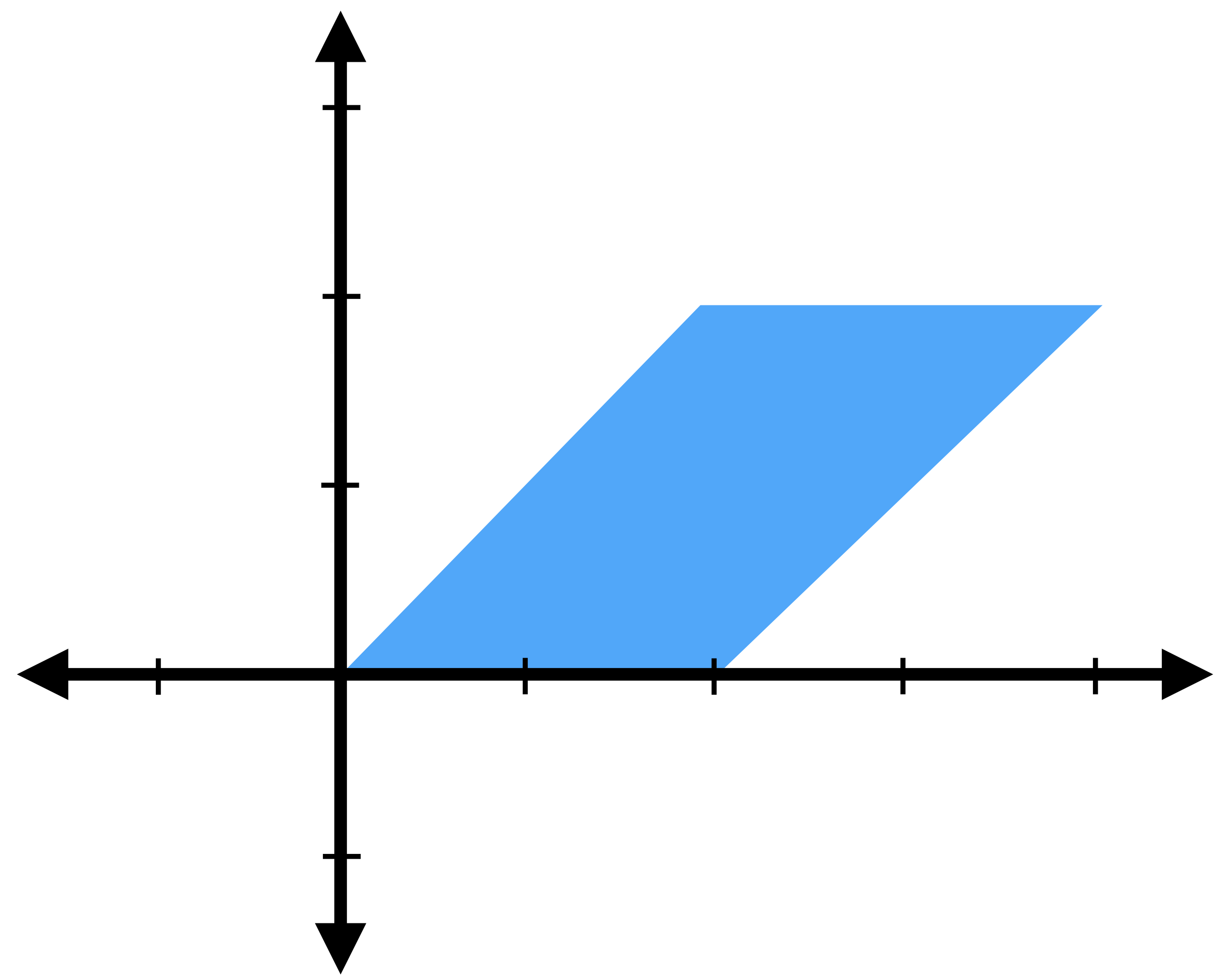
$$\mathbf{R}_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$





Shear



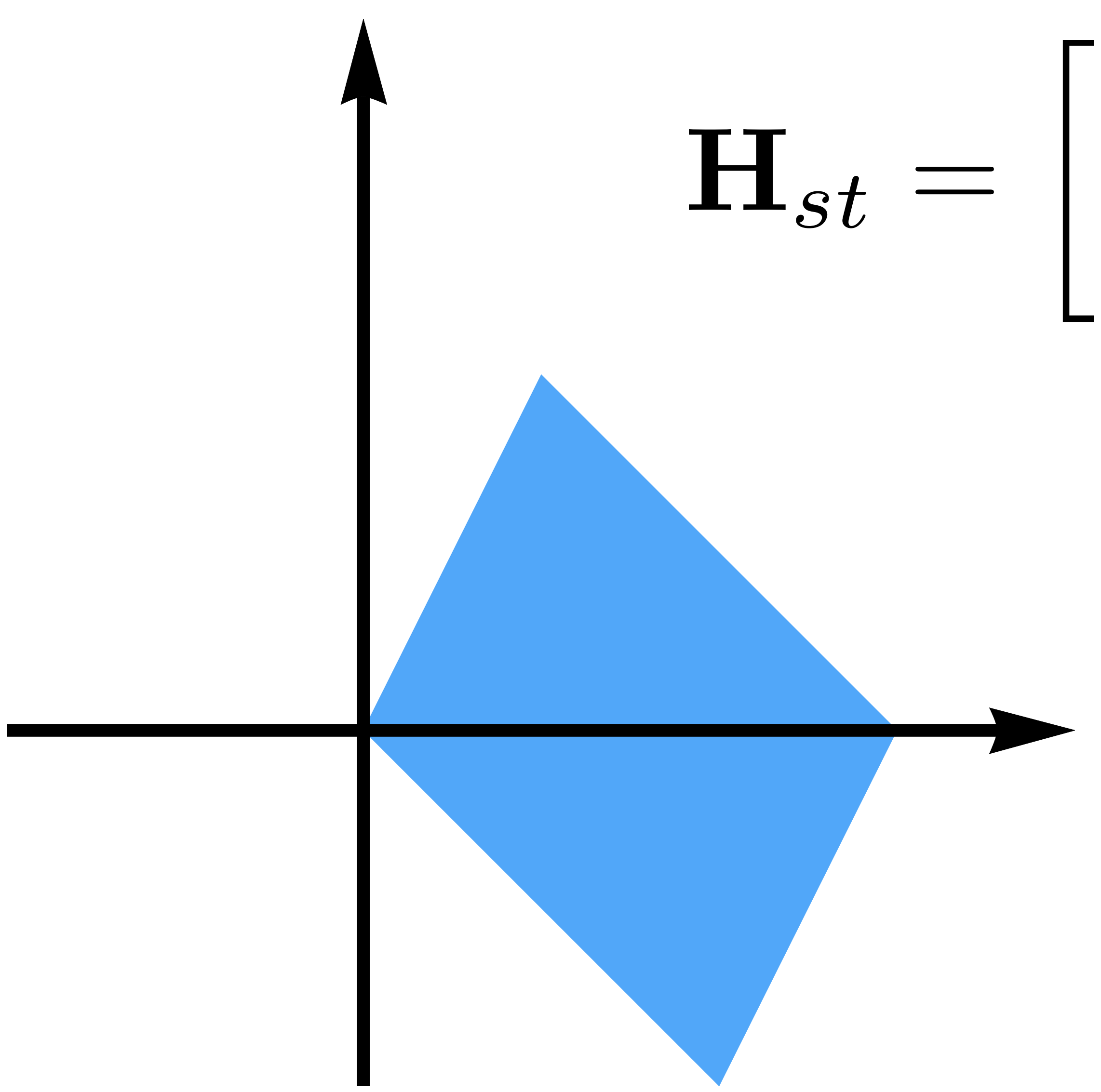


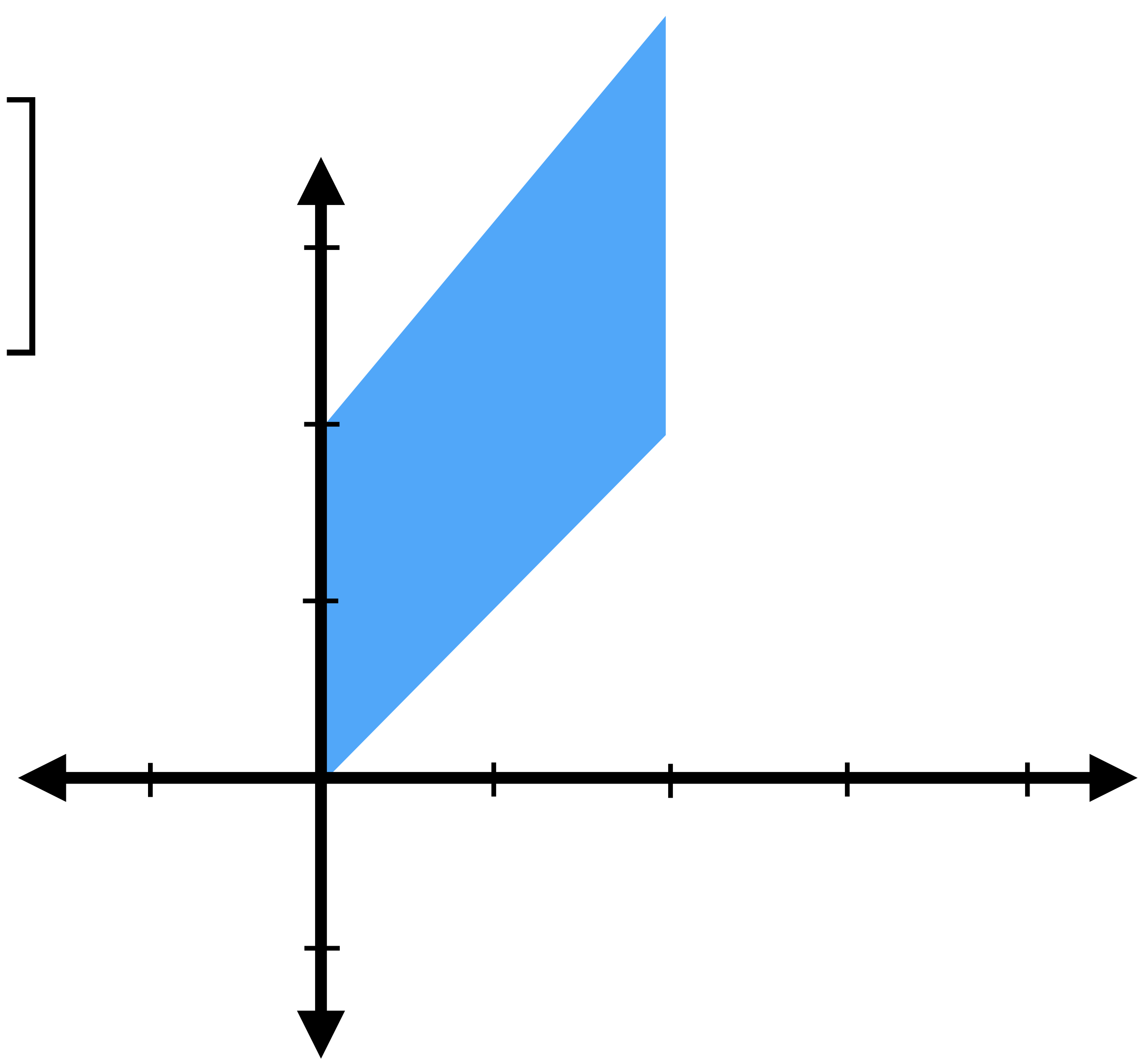
Shear in x:

$$\mathbf{H}_{xs} = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$$

Arbitrary shear:

$$\mathbf{H}_{st} = \begin{bmatrix} 1 & s \\ t & 1 \end{bmatrix}$$

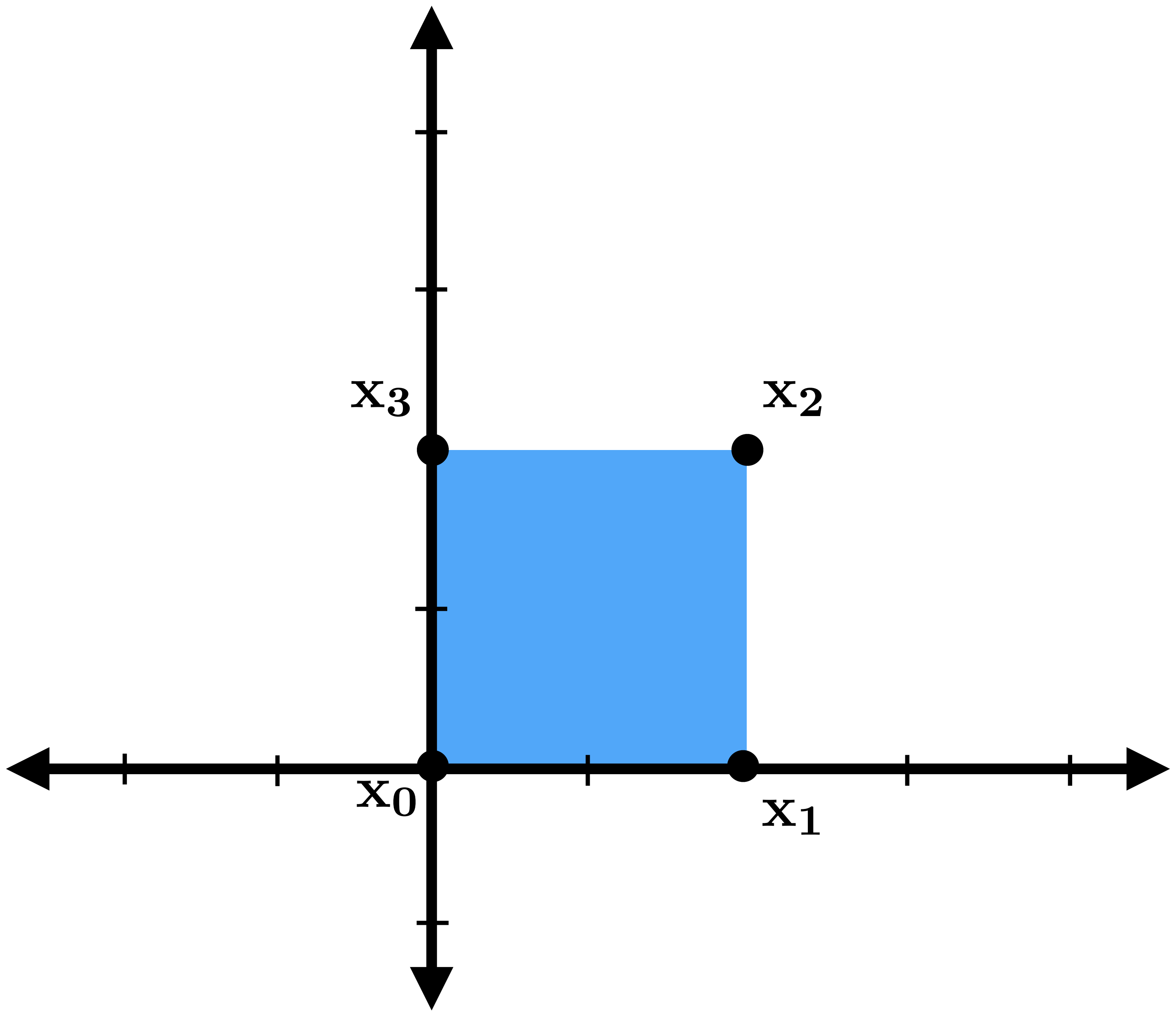


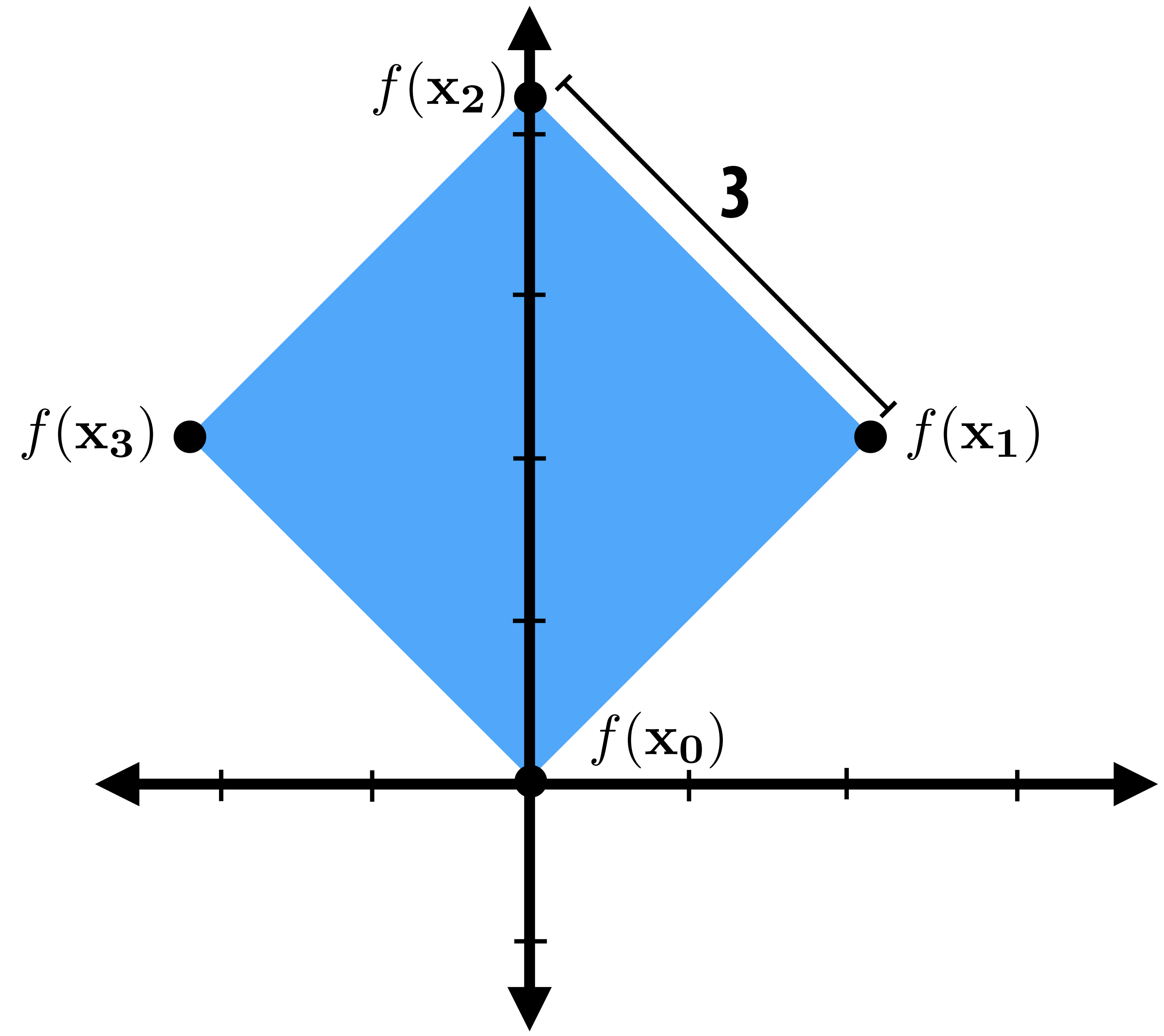


Shear in y:

$$\mathbf{H}_{ys} = \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix}$$

How do we compose linear transformations?





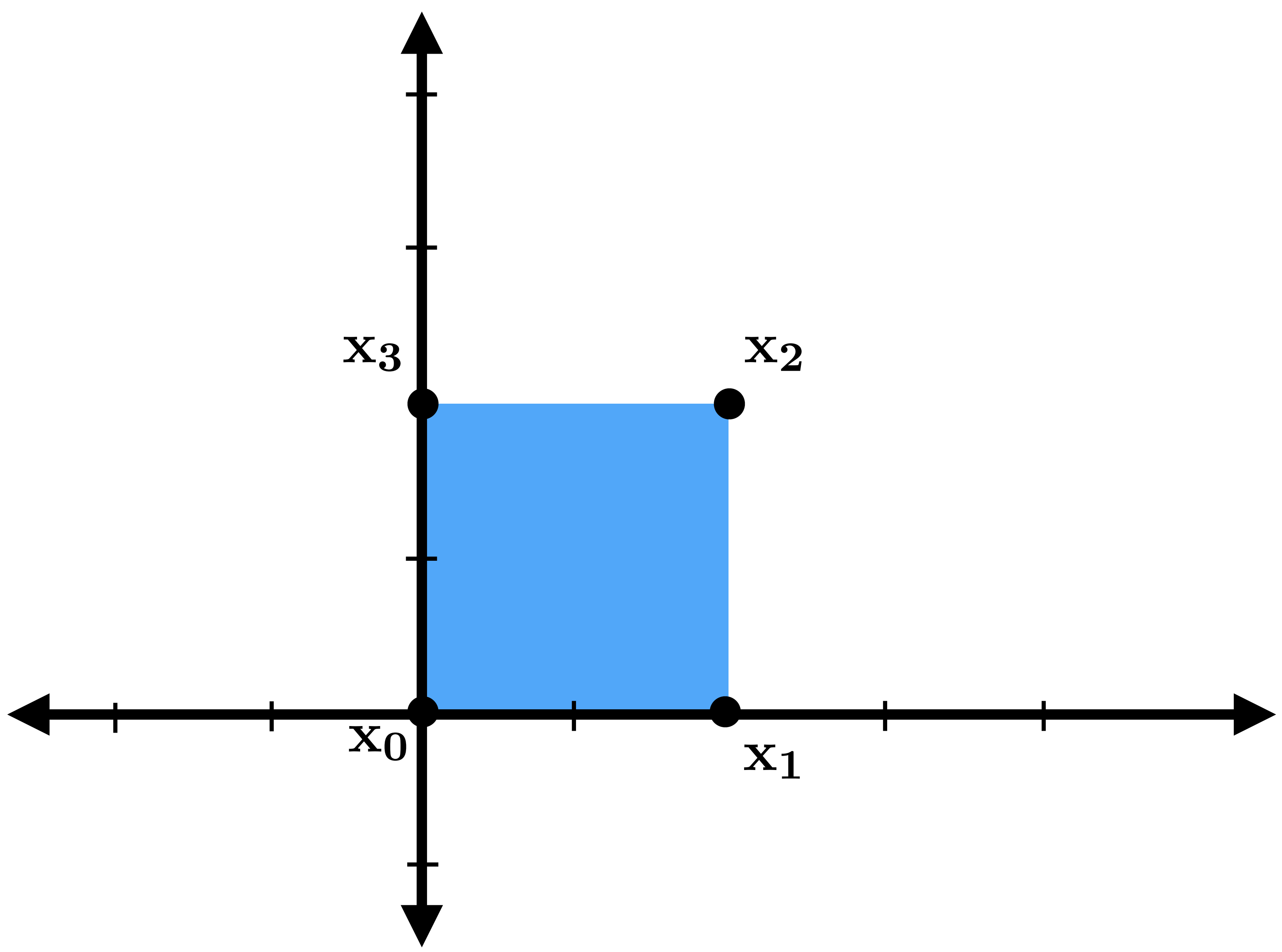
Compose linear transformations via matrix multiplication.
This example: uniform scale, followed by rotation

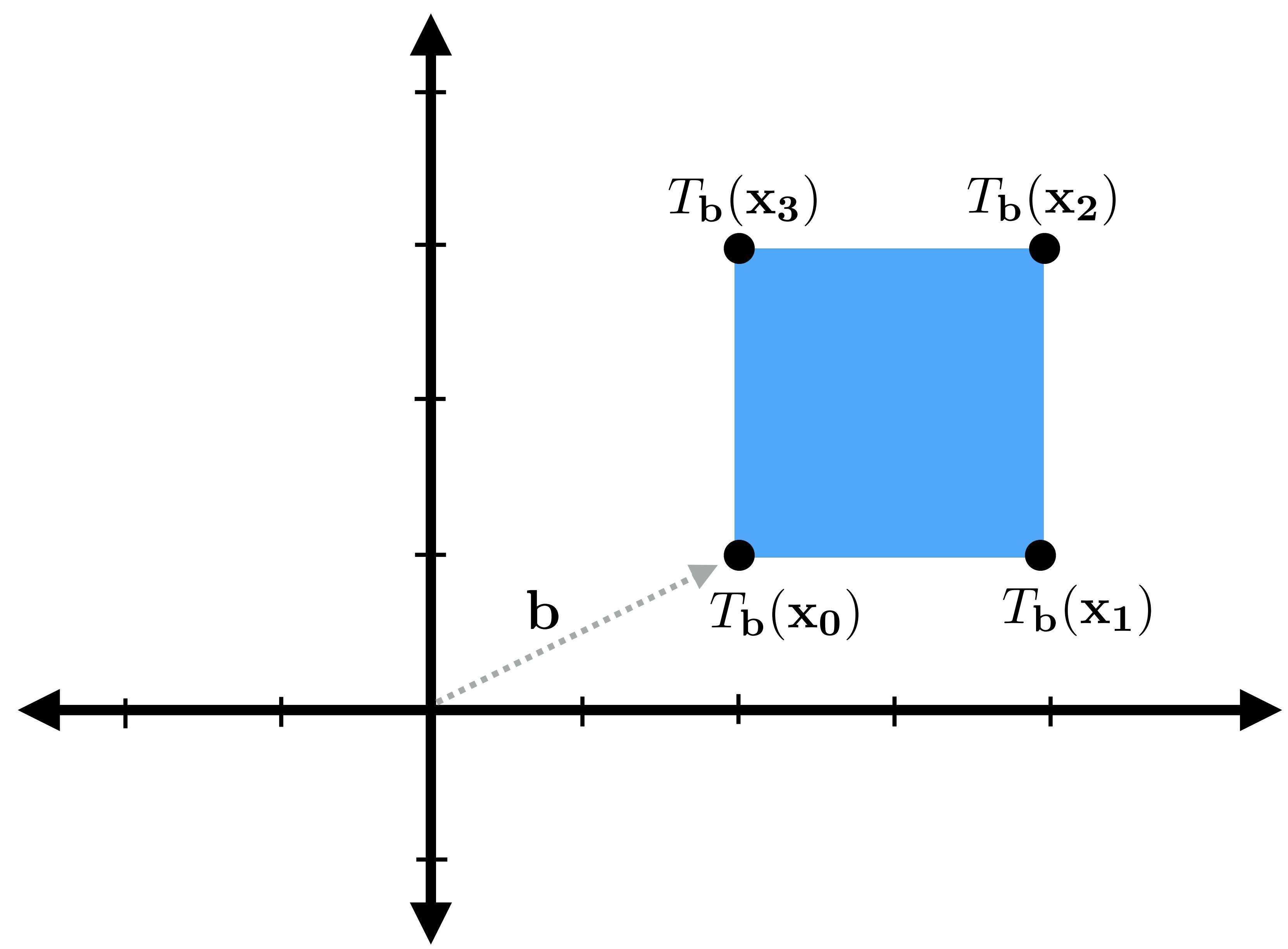
$$f(\mathbf{x}) = R_{\pi/4}\mathbf{S}_{[1.5,1.5]}\mathbf{x}$$

Enables simple, efficient implementation: reduce complex chain of transformations to a single matrix multiplication.

How do we deal with translation? (Not linear)

$$T_{\mathbf{b}}(\mathbf{x}) = \mathbf{x} + \mathbf{b}$$





Recall: translation is not a linear transform

→ Output coefficients are not a linear combination of input coefficients

→ Translation operation cannot be represented by a 2x2 matrix

$$\mathbf{x_{out}}_x = \mathbf{x}_x + \mathbf{b}_x$$

$$\mathbf{x_{out}}_y = \mathbf{x}_y + \mathbf{b}_y$$

Translation math

2D homogeneous coordinates (2D-H)

Idea: represent 2D points with THREE values ("homogeneous coordinates")

So the point (x, y) **is represented as the 3-vector:** $\begin{bmatrix} x & y & 1 \end{bmatrix}^T$

And transformations are represented a 3x3 matrices that transform these vectors.

Recover final 2D coordinates by dividing by "extra" (third) coordinate

$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} \Rightarrow \begin{bmatrix} x/w \\ y/w \end{bmatrix}$$

(More on this later…)

Example: scale and rotation in 2D-H coords

- **For transformations that are already linear, not much changes:**

$$\mathbf{S_s} = \begin{bmatrix} \mathbf{S}_x & 0 & 0 \\ 0 & \mathbf{S}_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{R}_\theta = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Notice that the last row/column doesn't do anything interesting. E.g., for scaling:

$$\mathbf{S_s} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{S}_x x \\ \mathbf{S}_y y \\ 1 \end{bmatrix}$$

Now we divide by the 3rd coordinate to get our final 2D coordinates (not too exciting!)

$$\begin{bmatrix} \mathbf{S}_x x \\ \mathbf{S}_y y \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} \mathbf{S}_x x/1 \\ \mathbf{S}_y y/1 \end{bmatrix} = \begin{bmatrix} \mathbf{S}_x x \\ \mathbf{S}_y y \end{bmatrix}$$

(Will get more interesting when we talk about *perspective*…)

Translation in 2D homogeneous coordinates

Translation expressed as 3x3 matrix multiplication:

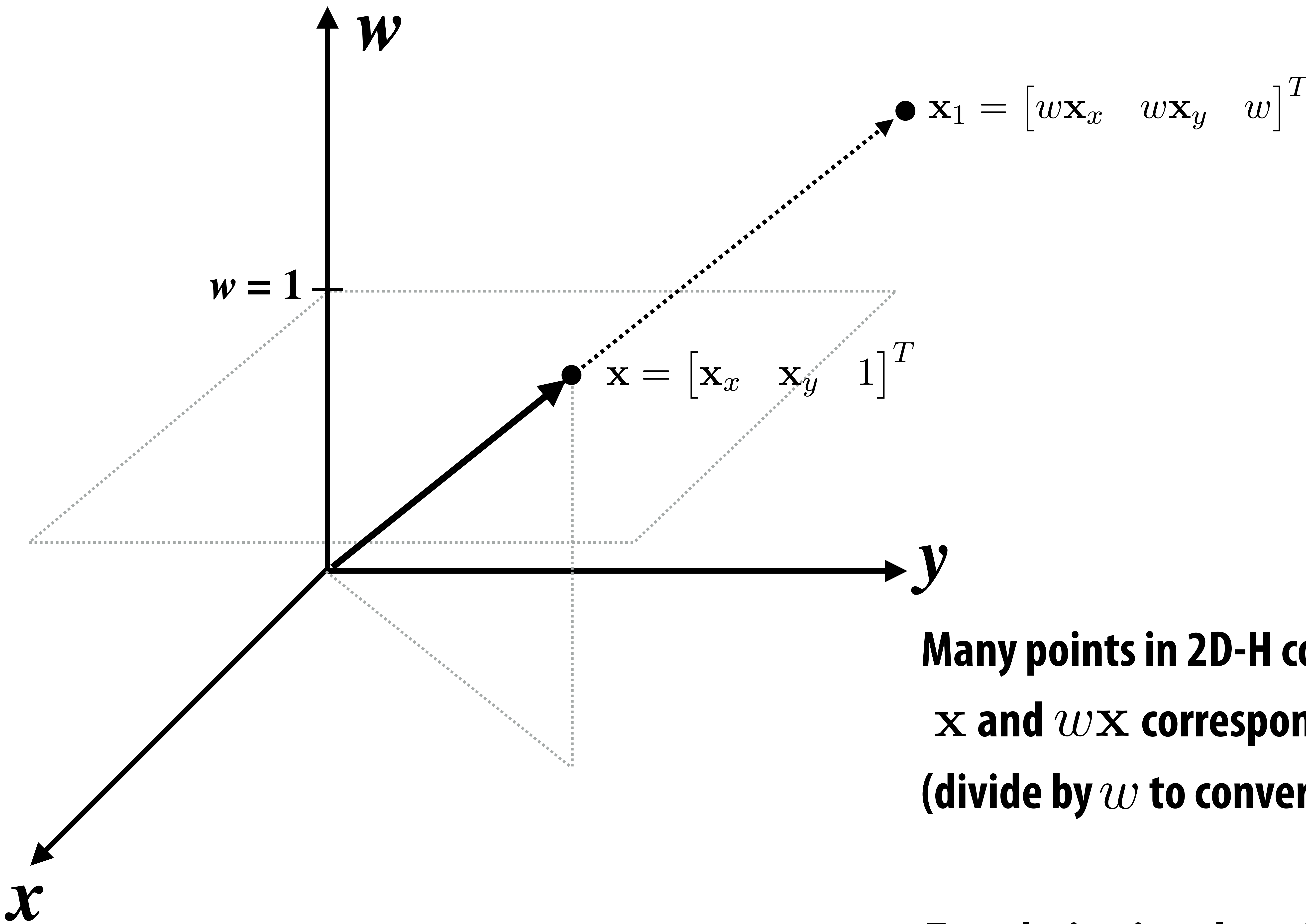
$$\mathbf{T_b} = \begin{bmatrix} 1 & 0 & \mathbf{b}_x \\ 0 & 1 & \mathbf{b}_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{T_b x} = \begin{bmatrix} 1 & 0 & \mathbf{b}_x \\ 0 & 1 & \mathbf{b}_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x}_x \\ \mathbf{x}_y \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{x}_x + \mathbf{b}_x \\ \mathbf{x}_y + \mathbf{b}_y \\ 1 \end{bmatrix}$$

(remember: just a linear combination of columns!)

Cool: homogeneous coordinates let us encode translations as *linear* transformations!

Homogeneous coordinates: some intuition



Many points in 2D-H correspond to same point in 2D

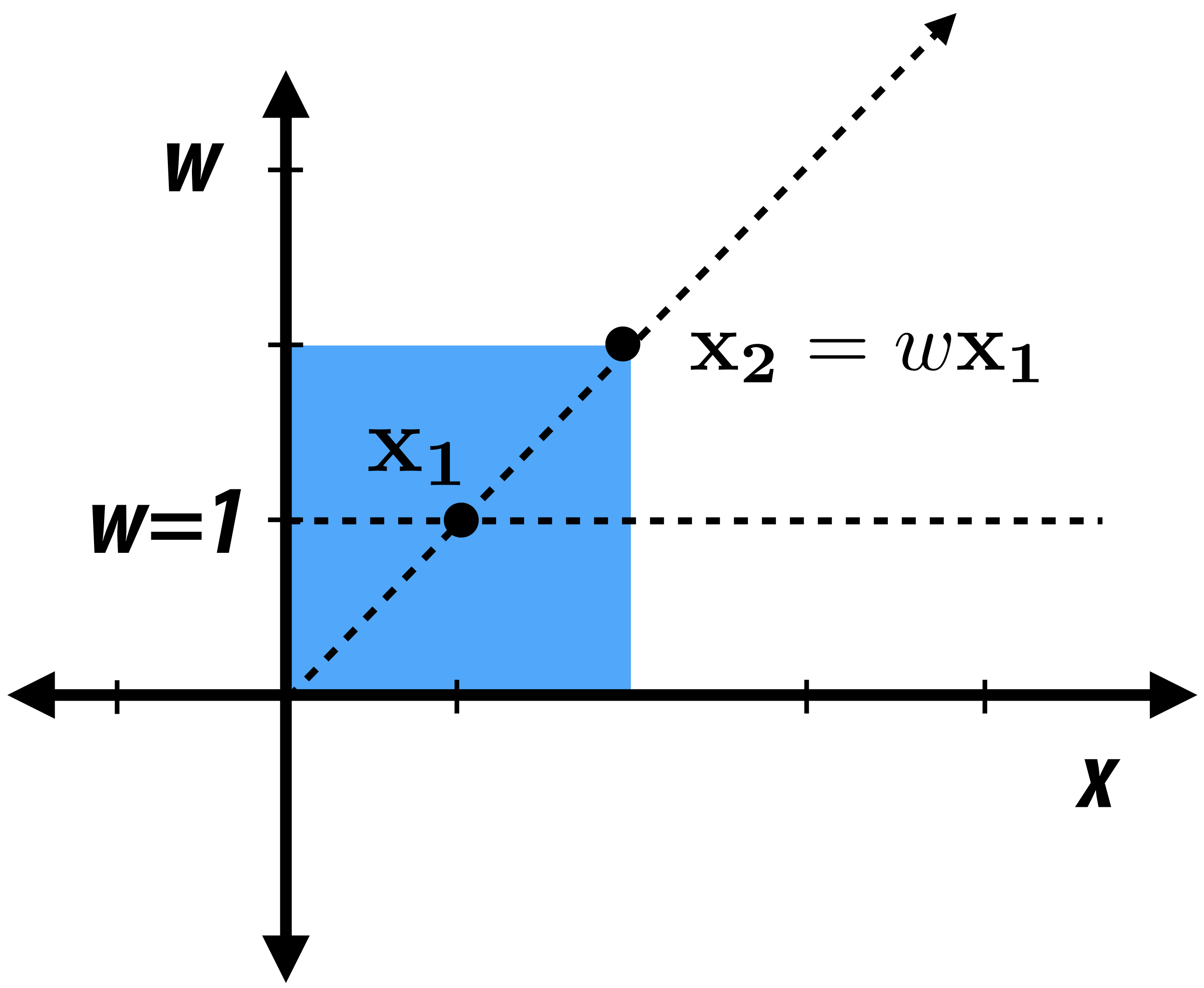
$\mathbf{x}$ **and** $w\mathbf{x}$ **correspond to the same 2D point**

(divide by w **to convert 2D-H back to 2D)**

Translation is a shear in x and y in 2D-H space

$$\mathbf{T_b}\mathbf{x} = \begin{bmatrix} 1 & 0 & \mathbf{b}_x \\ 0 & 1 & \mathbf{b}_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w\mathbf{x}_x \\ w\mathbf{x}_y \\ w \end{bmatrix} = \begin{bmatrix} w\mathbf{x}_x + w\mathbf{b}_x \\ w\mathbf{x}_y + w\mathbf{b}_y \\ w \end{bmatrix}$$

Translation = shear in homogeneous space

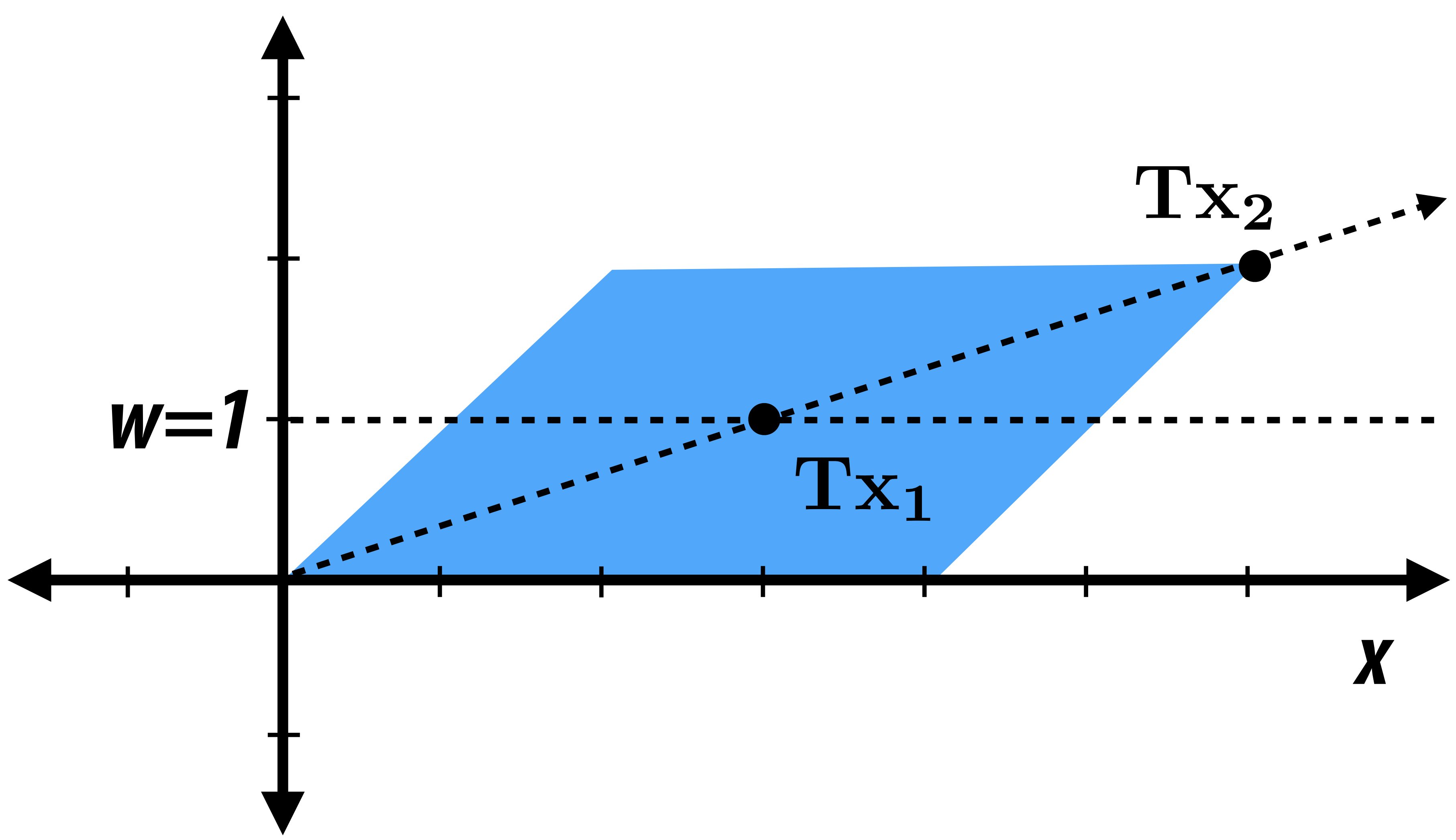


For simplicity, consider 1D-H:

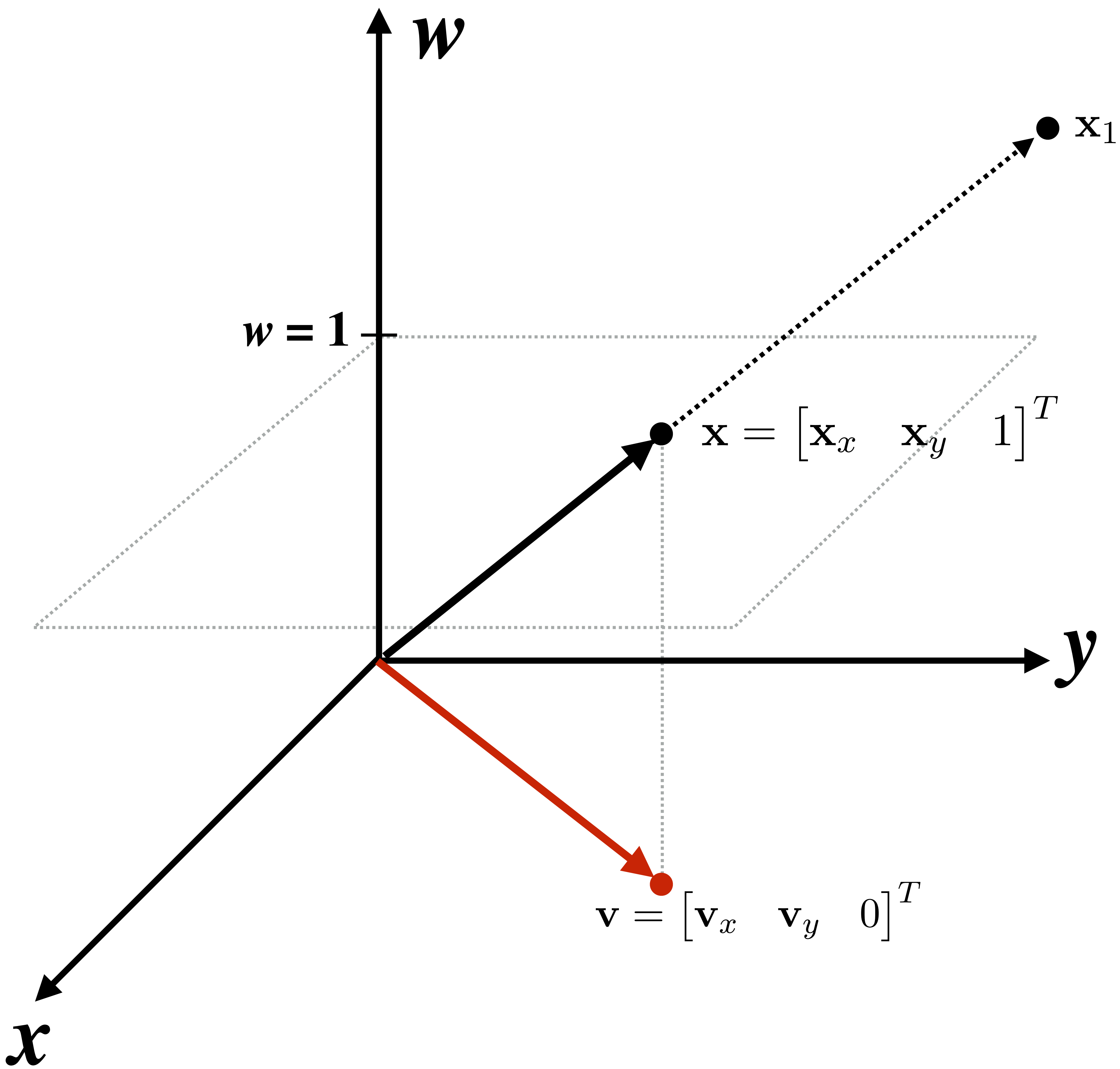
Translate by *t=2:* $\mathbf{T} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

Recall: this is a shear in homogeneous *x*.

1D translation is affine in 1D (*x* + *t*), but it is linear in 1D-H



Homogeneous coordinates: points vs. vectors



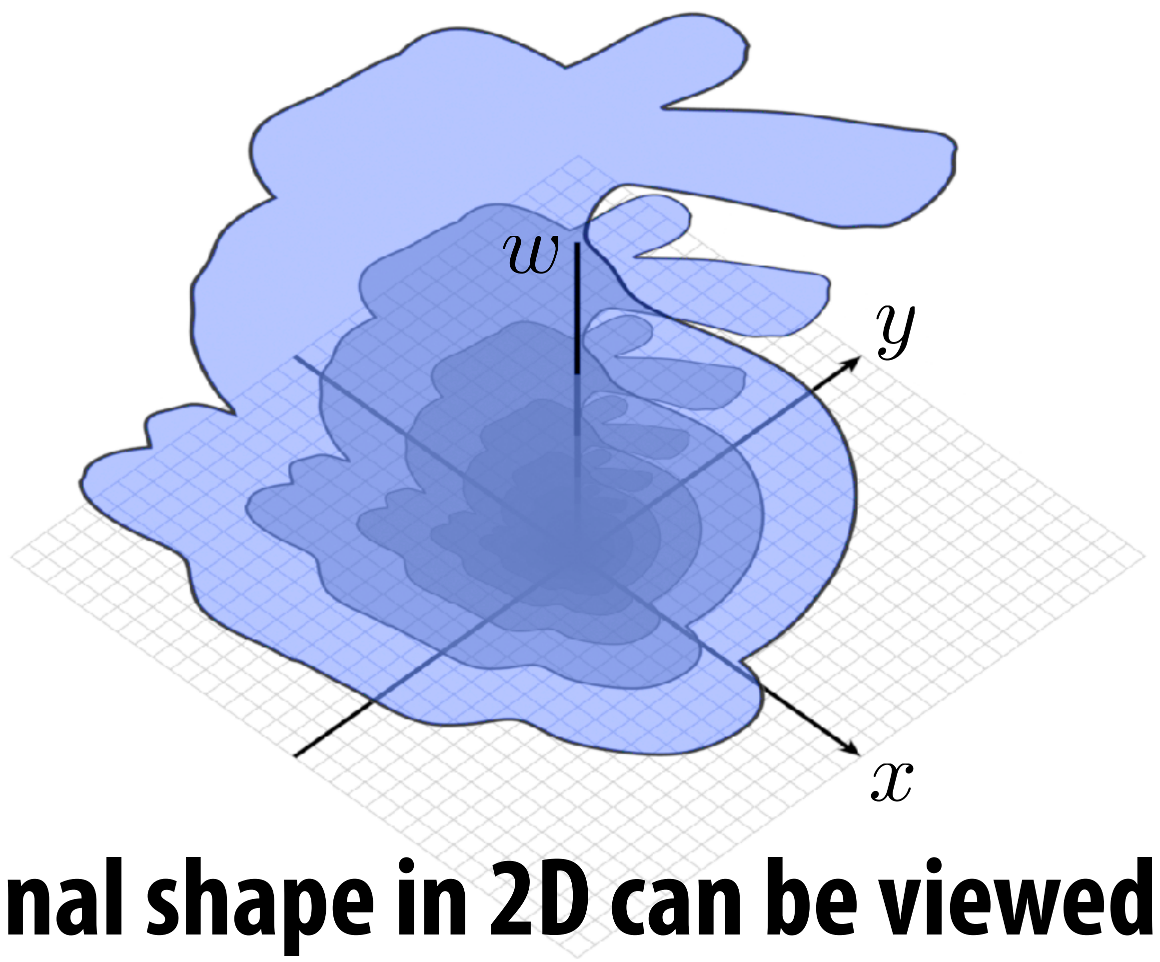
2D-H points with w=0 represent 2D vectors (think: directions are points at infinity)

Unlike 2D, points and directions are distinguishable by their representation in 2D-H

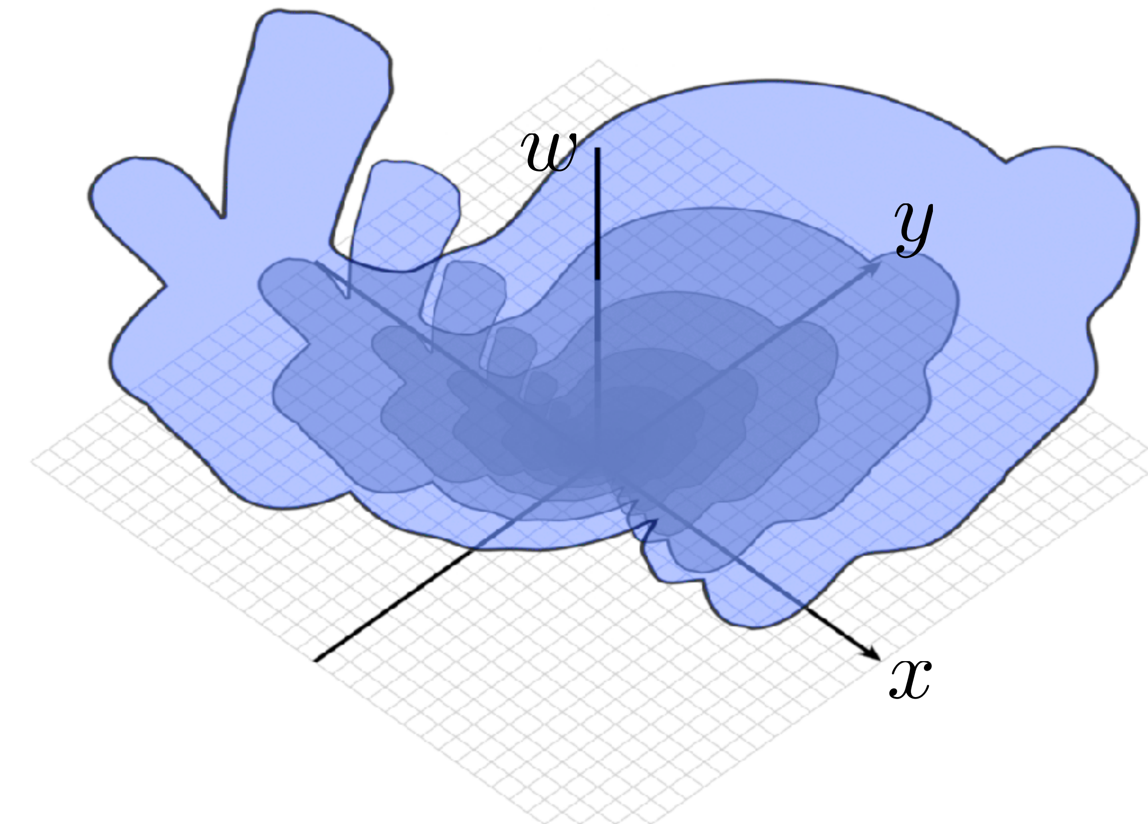
Note: translation does not modify directions:

$$\mathbf{T_b v} = \begin{bmatrix} 1 & 0 & \mathbf{b}_x \\ 0 & 1 & \mathbf{b}_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{v}_x \\ \mathbf{v}_y \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_x \\ \mathbf{v}_y \\ 0 \end{bmatrix}$$

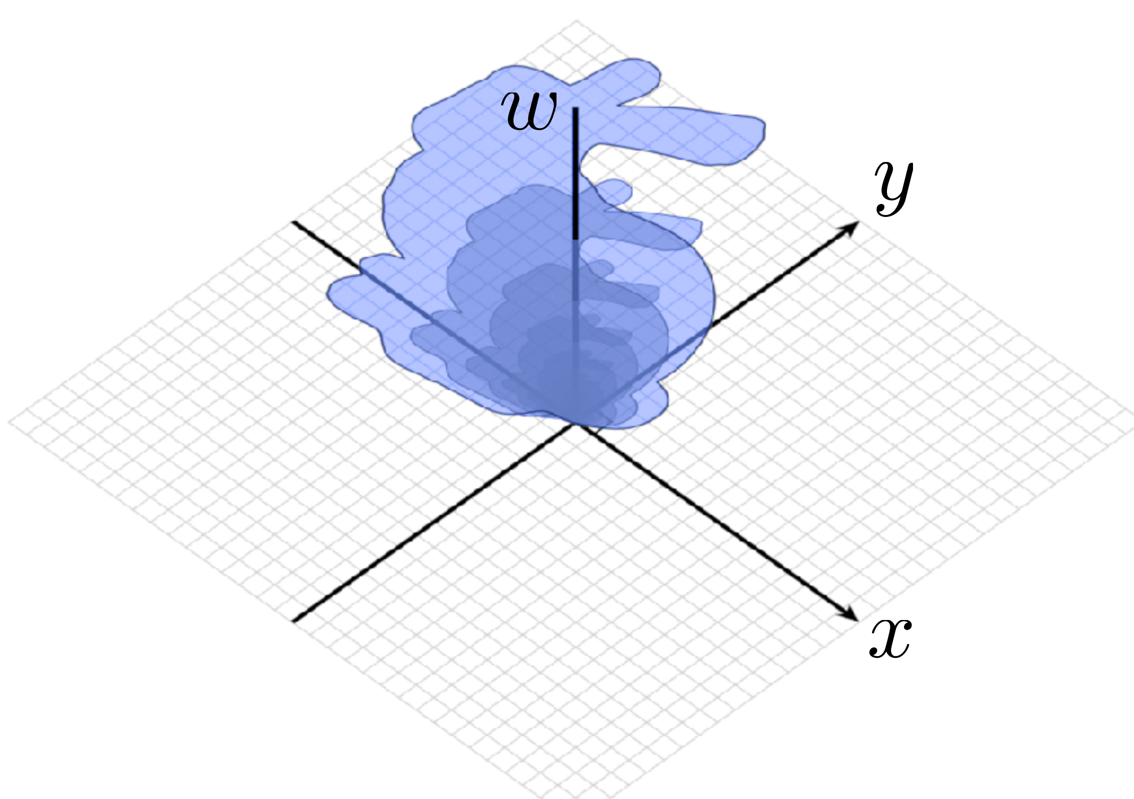
Visualizing 2D transformations in 2D-H



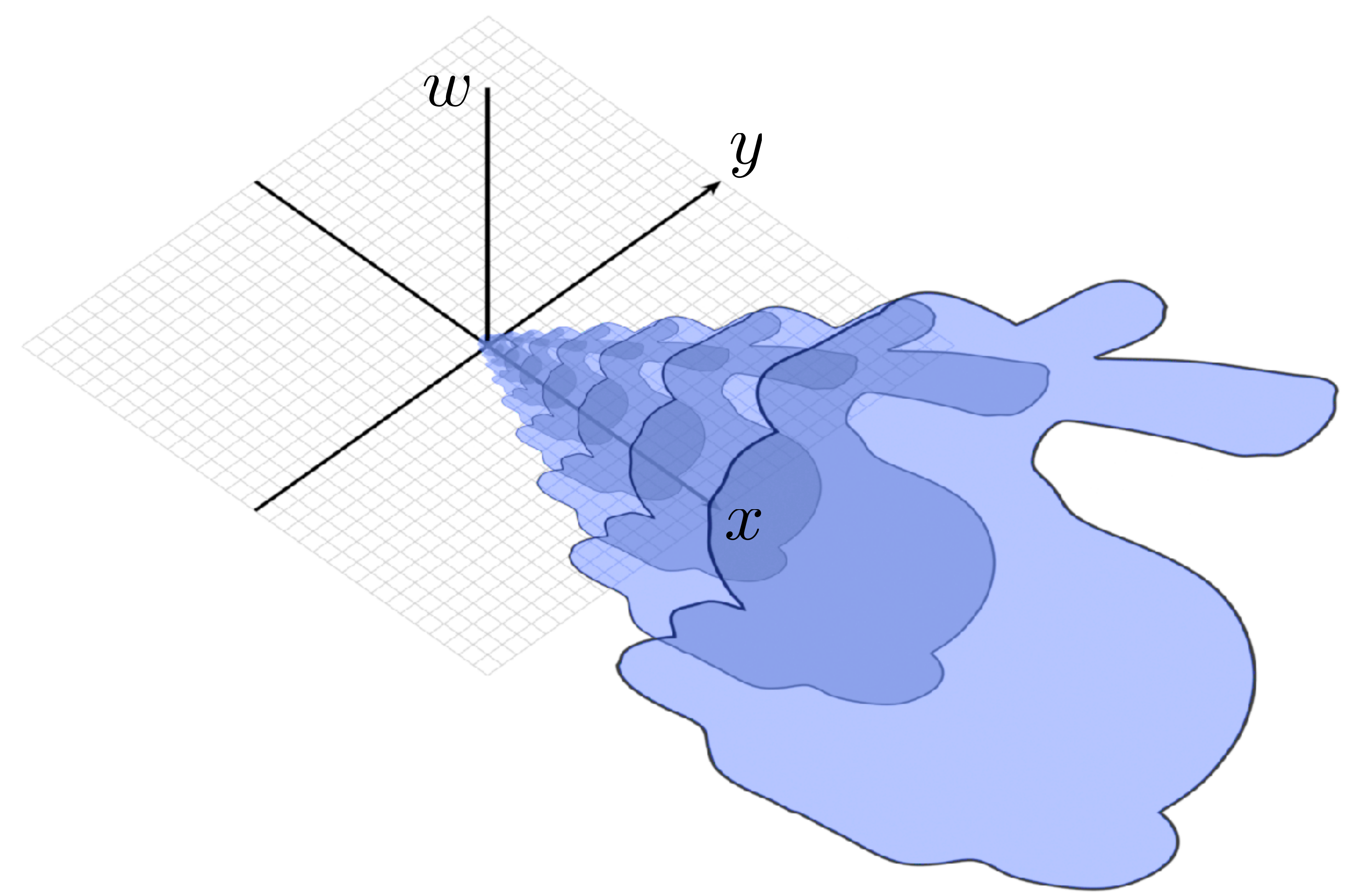
Original shape in 2D can be viewed as many copies, uniformly scaled by w.



2D rotation ↔ rotate around w



2D scale ↔ scale x and y; preserve w
(Question: what happens to 2D shape if you scale x, y, and w uniformly?)



2D translate ↔ shear in 2D-H
(LINEAR!)

Moving to 3D (and 3D-H)

Represent 3D transformations as 3x3 matrices and 3D-H transformations as 4x4 matrices

Scale:

3D

$$\mathbf{S_s} = \begin{bmatrix} \mathbf{S}_x & 0 & 0 \\ 0 & \mathbf{S}_y & 0 \\ 0 & 0 & \mathbf{S}_z \end{bmatrix}$$

3D-H

$$\mathbf{S_s} = \begin{bmatrix} \mathbf{S}_x & 0 & 0 & 0 \\ 0 & \mathbf{S}_y & 0 & 0 \\ 0 & 0 & \mathbf{S}_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Shear (in x, based on y,z position):

$$\mathbf{H}_{x,\mathbf{d}} = \begin{bmatrix} 1 & \mathbf{d}_y & \mathbf{d}_z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{H}_{x,\mathbf{d}} = \begin{bmatrix} 1 & \mathbf{d}_y & \mathbf{d}_z & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

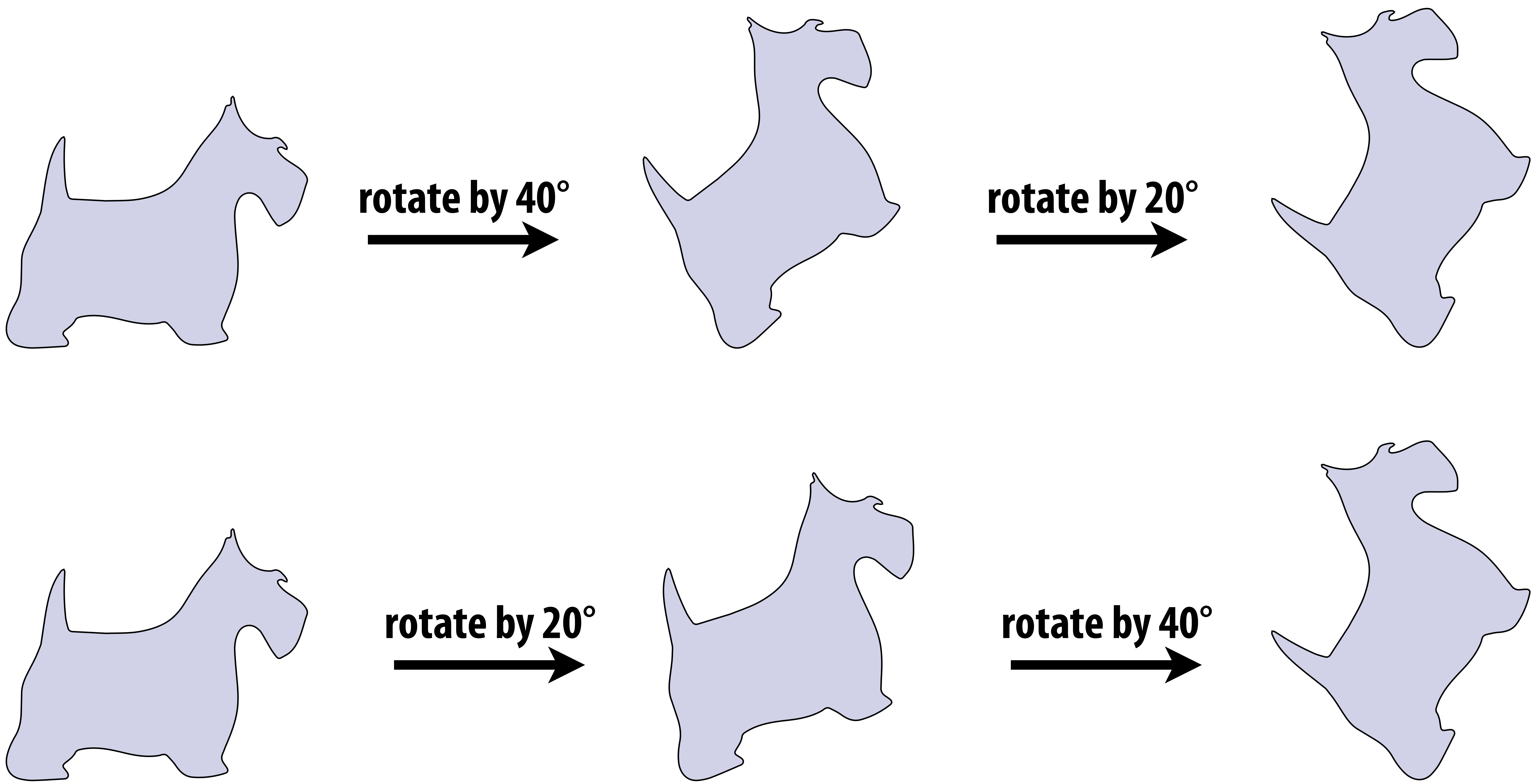
Translate:

3D-H

$$\mathbf{T_b} = \begin{bmatrix} 1 & 0 & 0 & \mathbf{b}_x \\ 0 & 1 & 0 & \mathbf{b}_y \\ 0 & 0 & 1 & \mathbf{b}_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Commutativity of rotations—2D

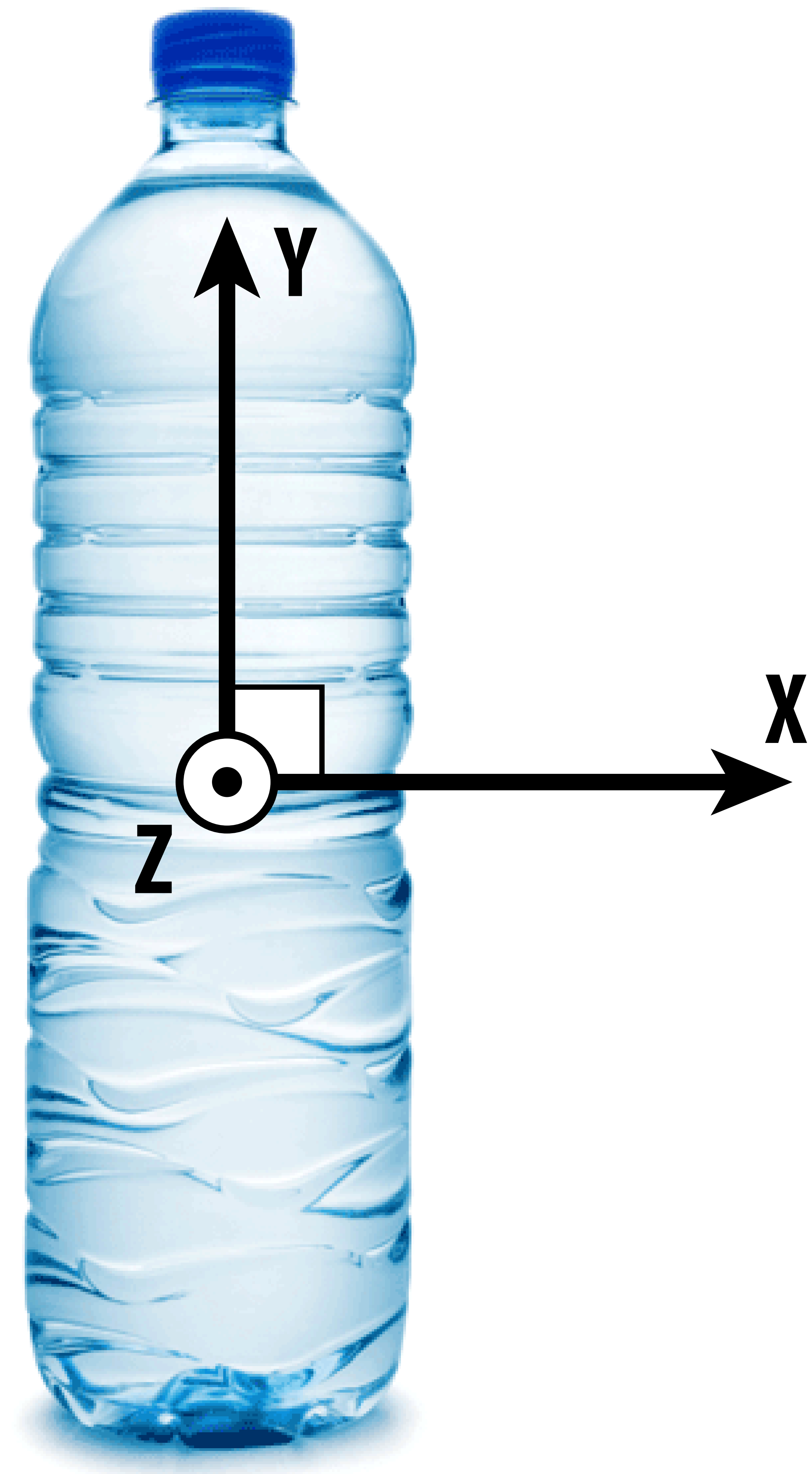
- **In 2D, order of rotations doesn't matter:**



Same result! ("2D rotations commute")

Commutativity of rotations—3D

- What about in 3D?
- IN-CLASS ACTIVITY:
  - Rotate 90° around Y, then 90° around Z, then 90° around X
  - Rotate 90° around Z, then 90° around Y, then 90° around X
  - (Was there any difference?)

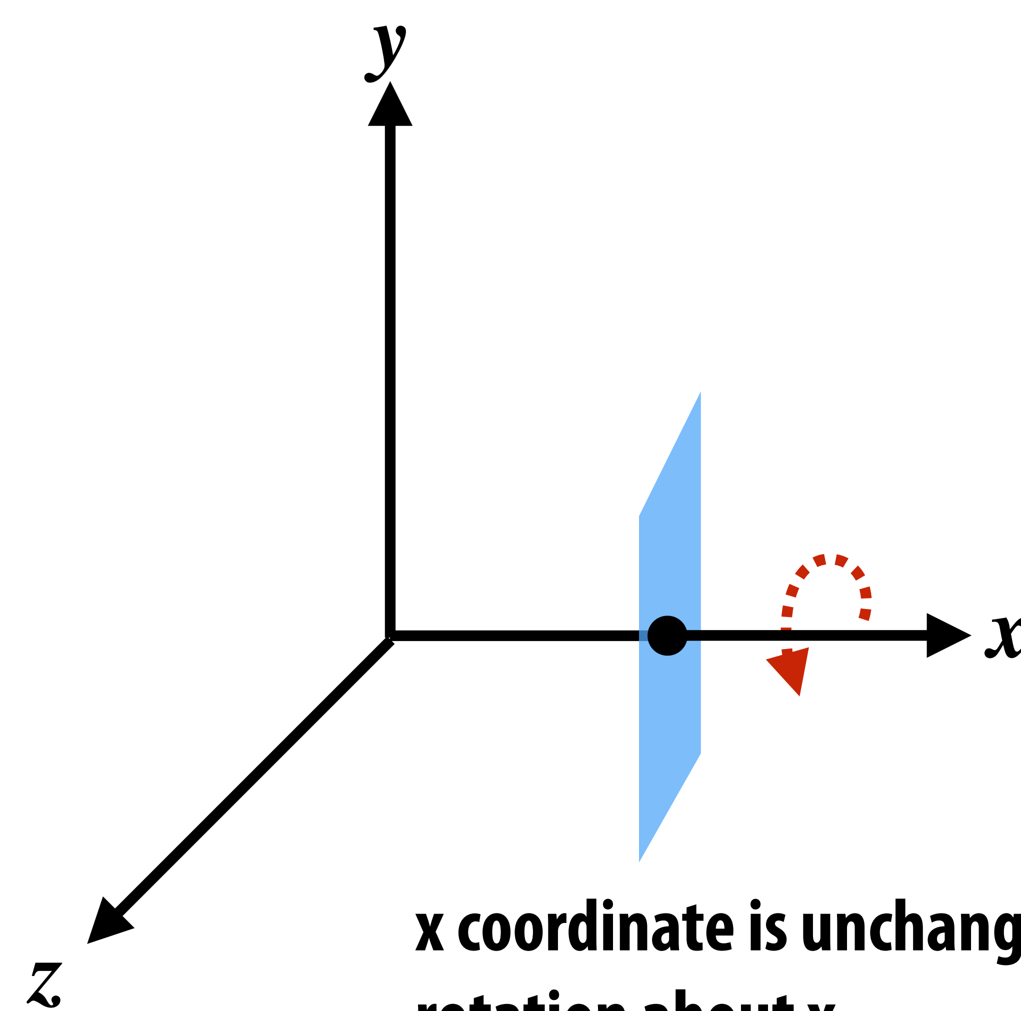


CONCLUSION: bad things can happen if we're not careful about the order in which we apply rotations!

Rotations in 3D

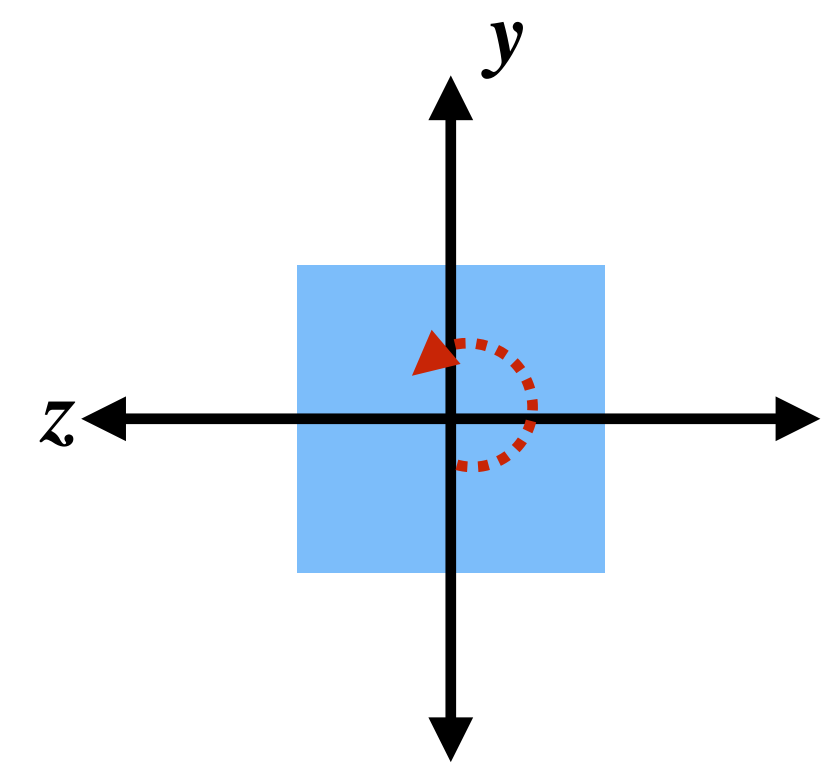
Rotation about x axis:

$$\mathbf{R}_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$



x coordinate is unchanged by rotation about x

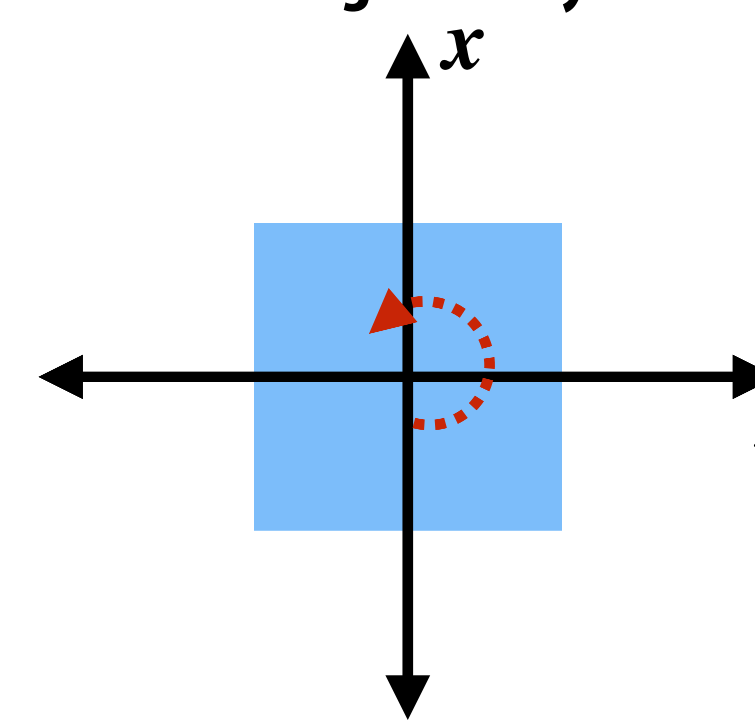
View looking down -x axis:



Rotation about y axis:

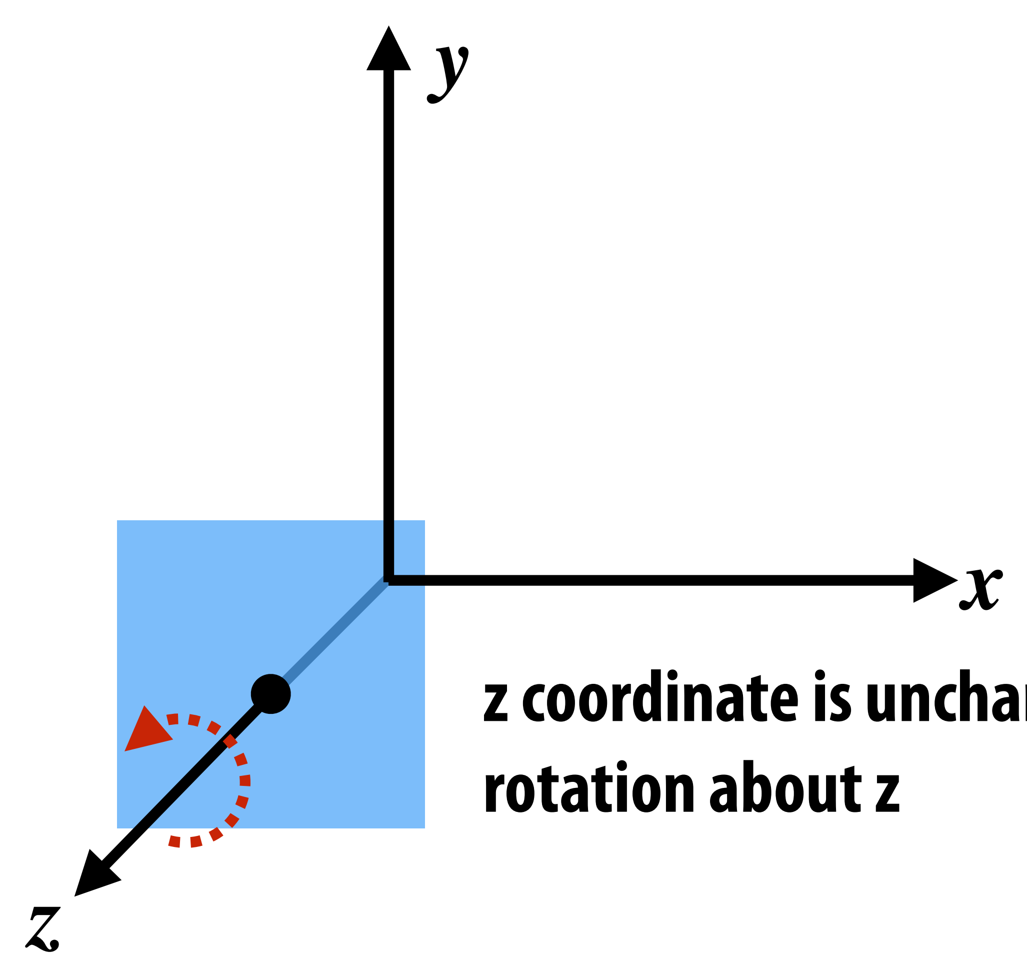
$$\mathbf{R}_{y,\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

View looking down -y axis:



Rotation about z axis:

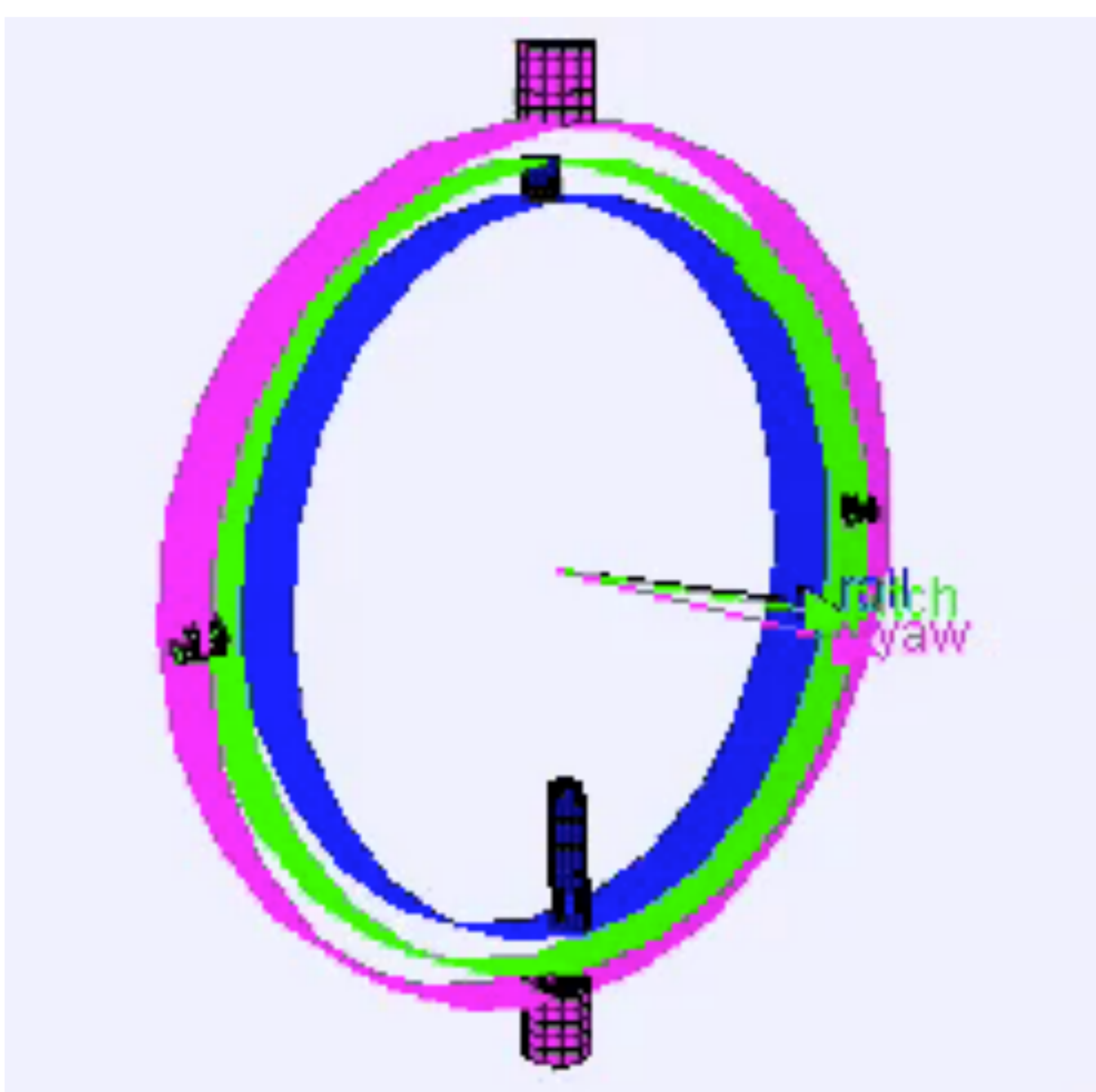
$$\mathbf{R}_{z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

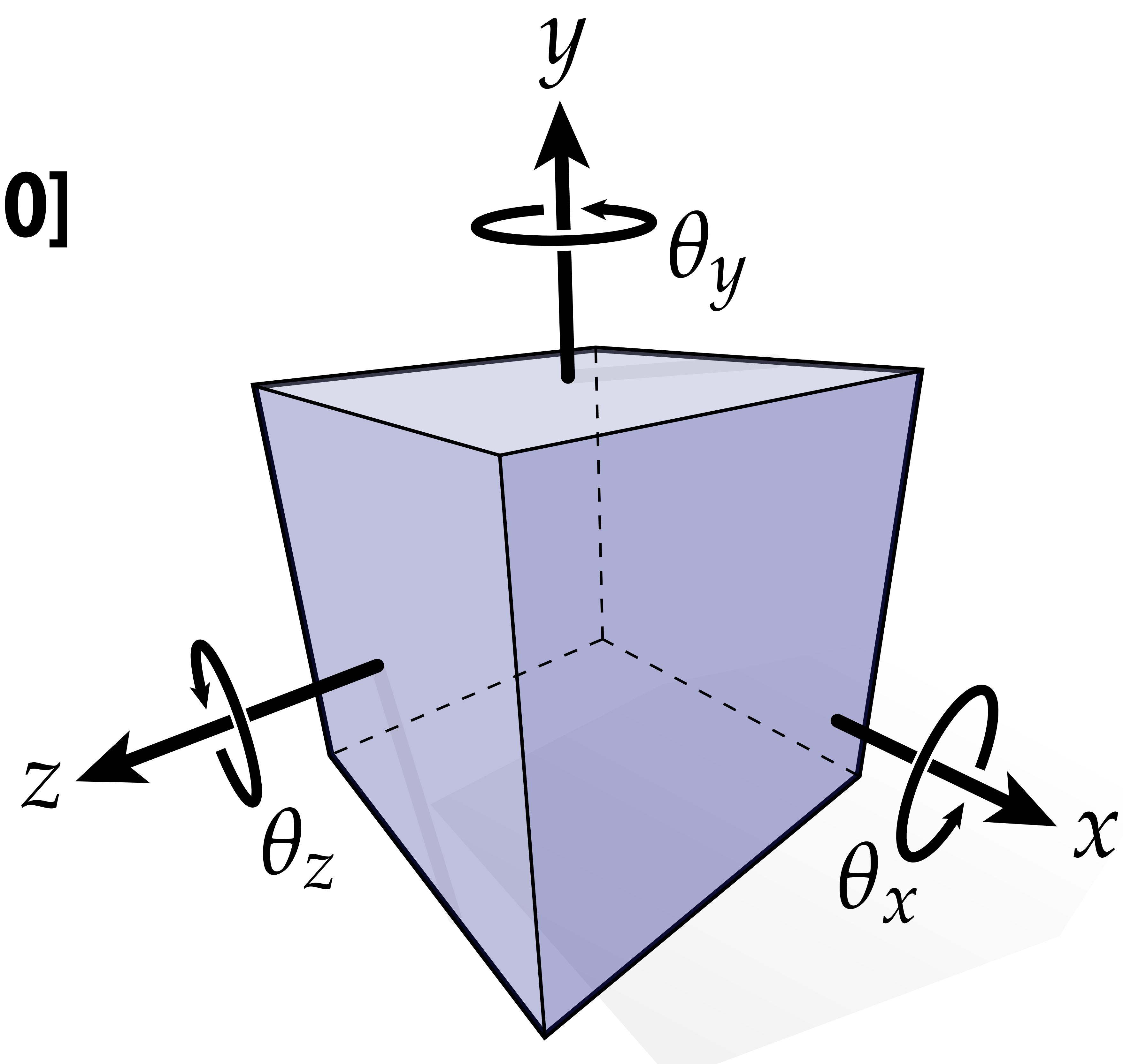


z coordinate is unchanged by rotation about z

Representing rotations in 3D—euler angles

- How do we express rotations in 3D?
- One idea: we know how to do 2D rotations
- Why not simply apply rotations around the three axes? (X,Y,Z)
- Scheme is called *Euler angles*
- PROBLEM: "Gimbal Lock" [DEMO]





Alternative representations of 3D rotations

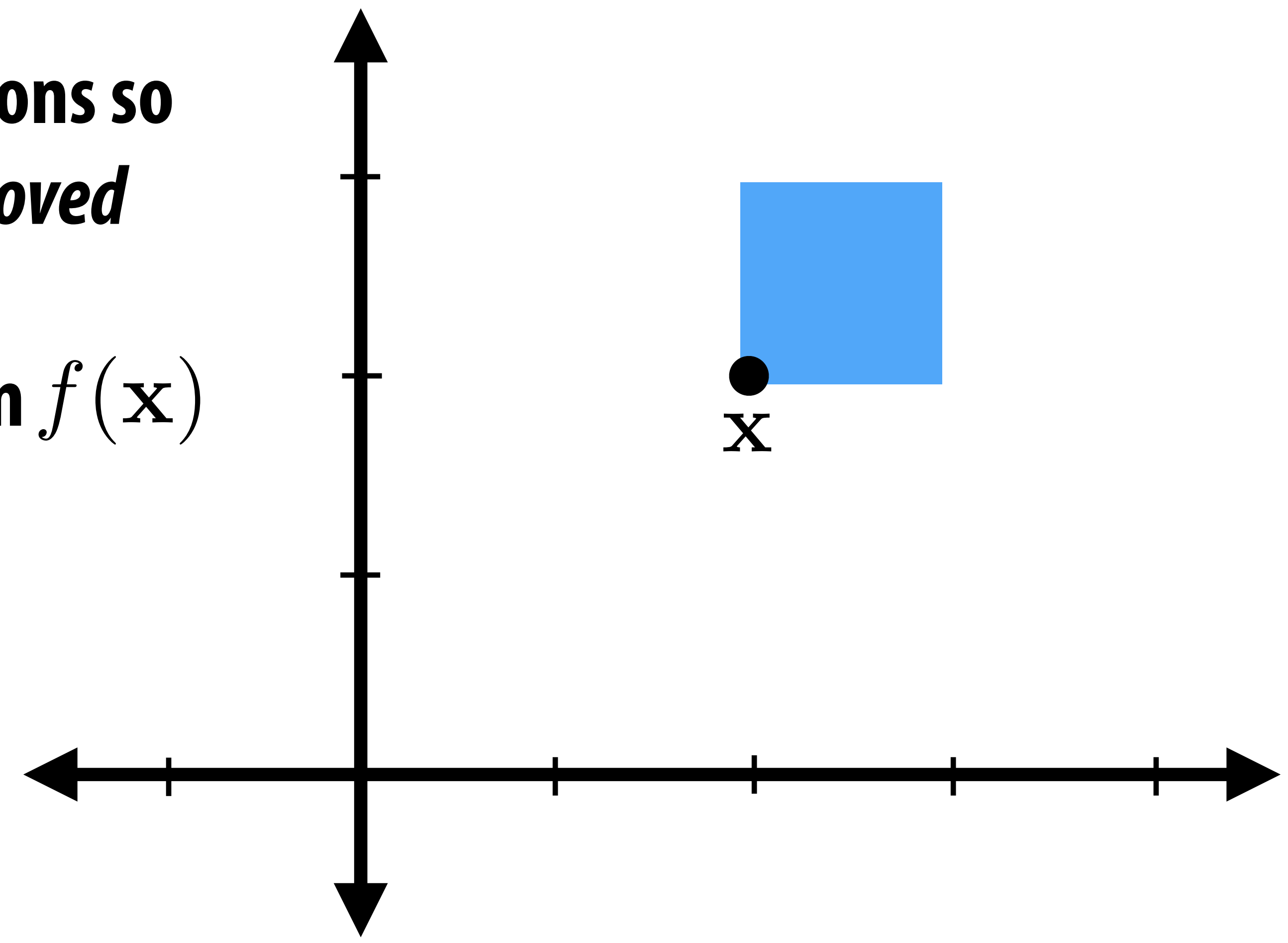
- Axis-angle rotations

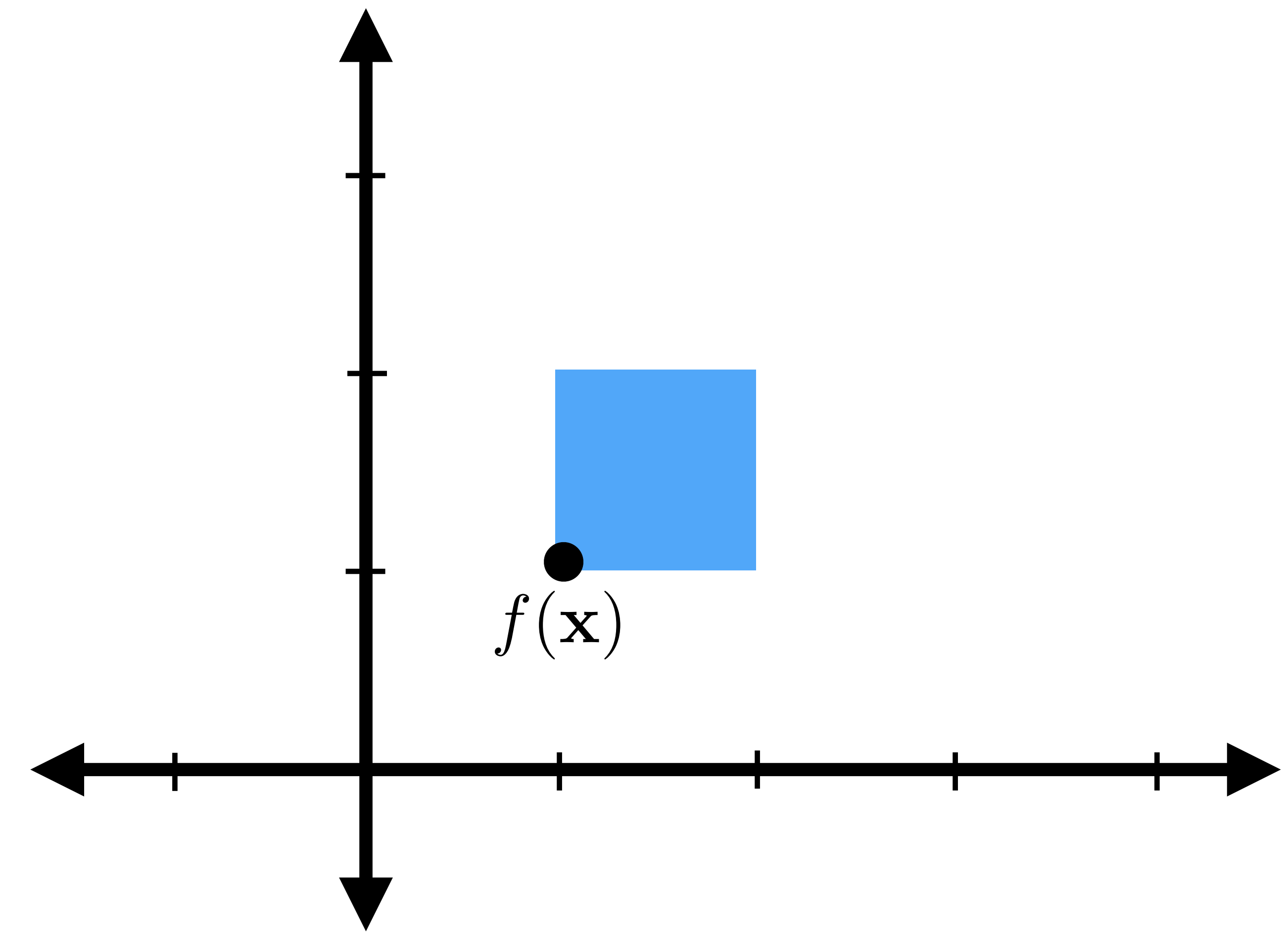
- Quaternions (not today)

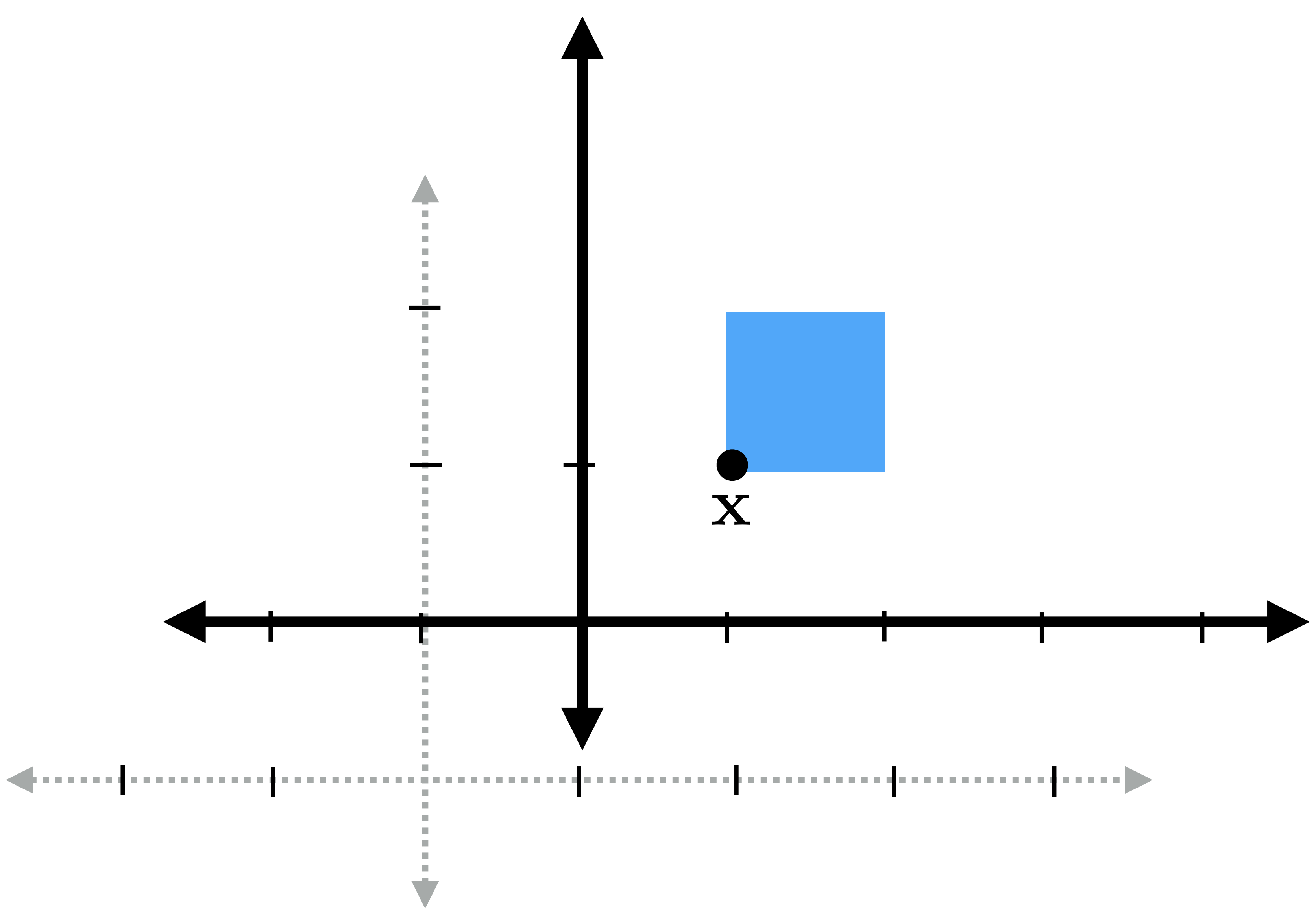
Another way to think about transformations: change of coordinates

Interpretation of transformations so far in this lecture: ***points get moved***

Point $\mathbf{x}$ moved to new position $f(\mathbf{x})$







Alternative interpretation:

Transformations induce of change of coordinate frame: Representation of $\mathbf{x}$ changes since point is now expressed in new coordinates

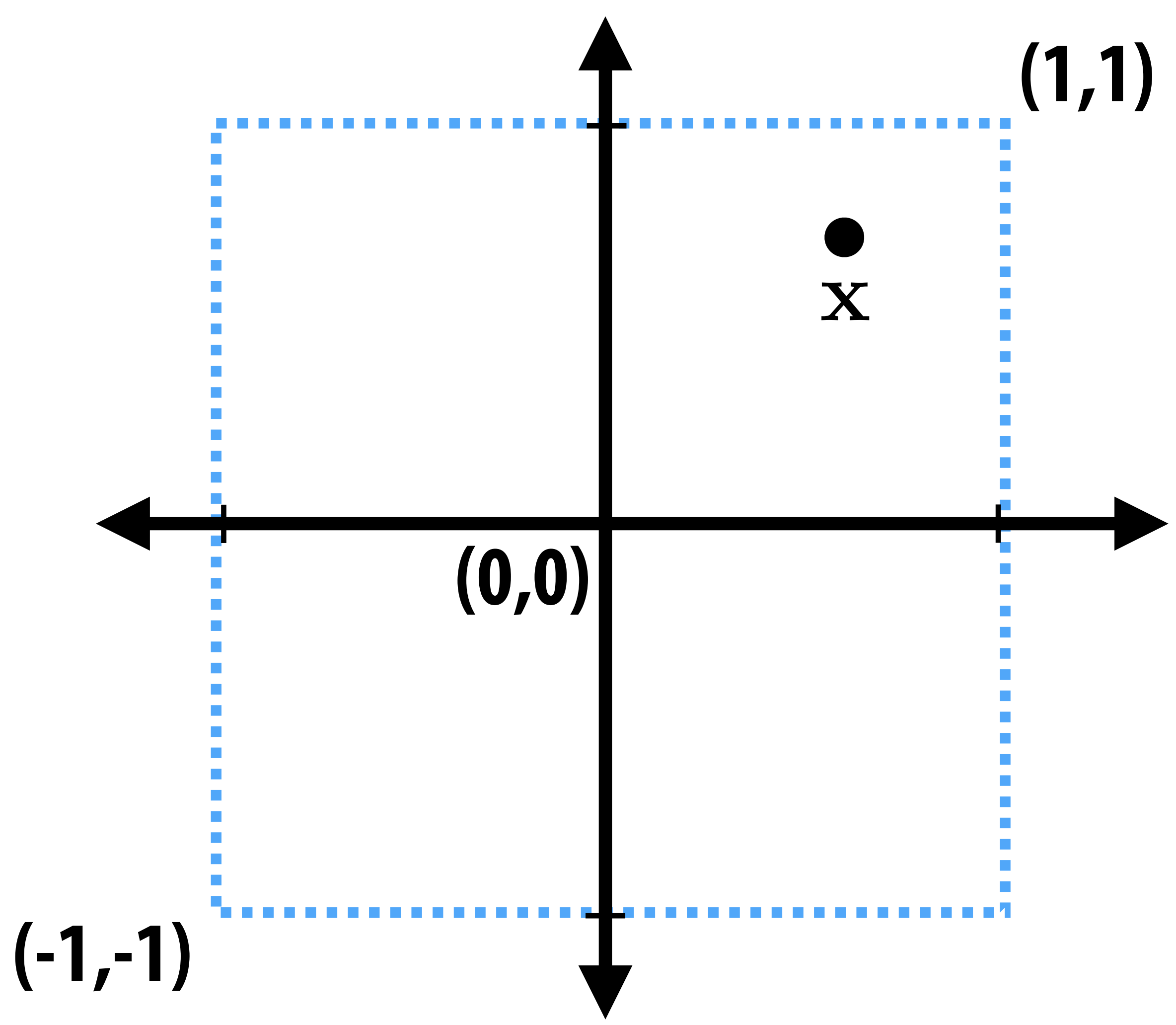
Screen transformation *

Convert points in normalized coordinate space to screen pixel coordinates

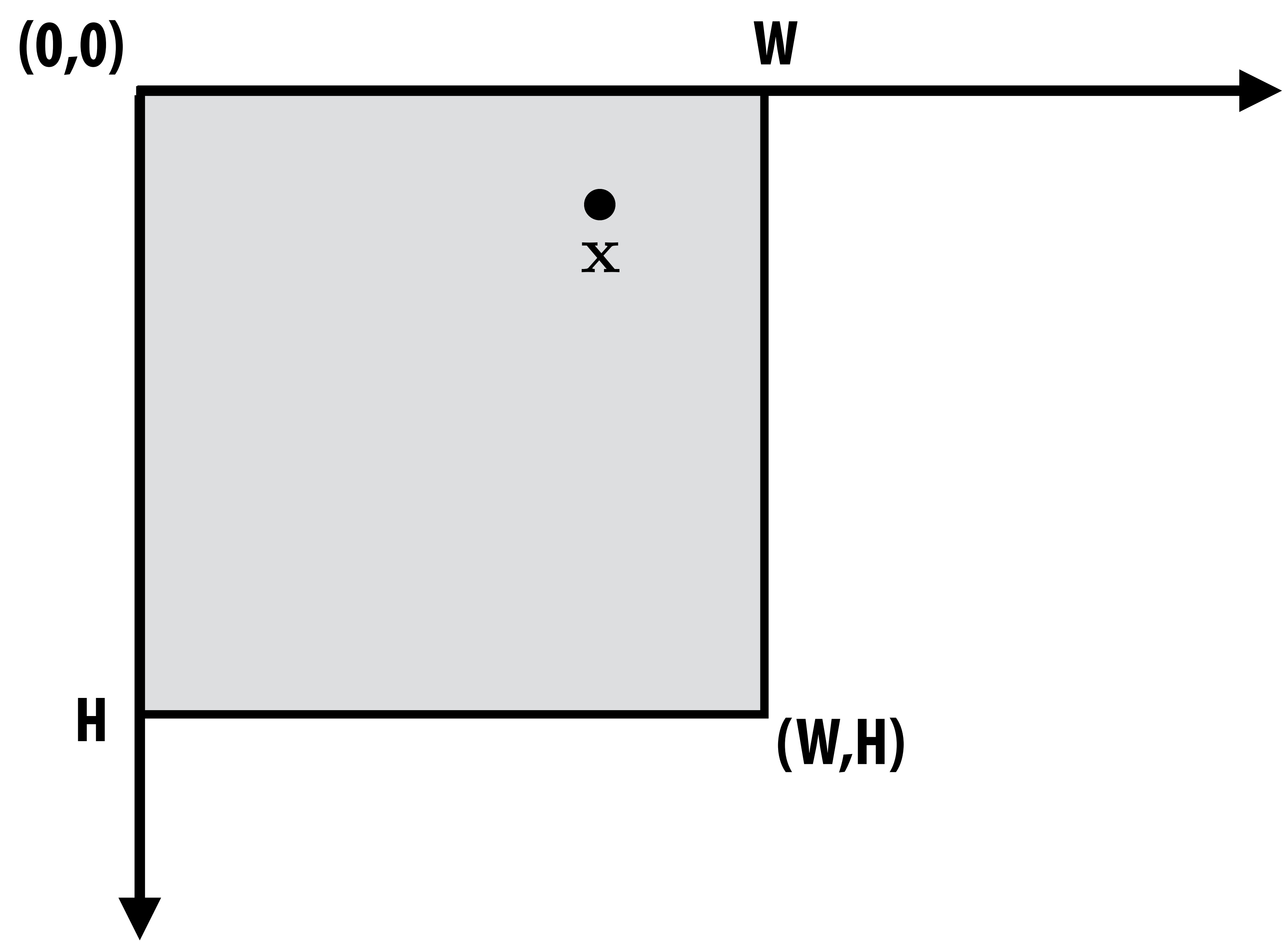
Example: all points within (-1,1) to (1,1) region are on screen

- **(1,1) in normalized space maps to (W,0) in screen space**
- **(-1,-1) in normalized space maps to (0,H) in screen space**

Normalized coordinate space:



Screen (W x H output image) coordinate space:



Step 1: reflect about x

Step 2: translate by (1,1)

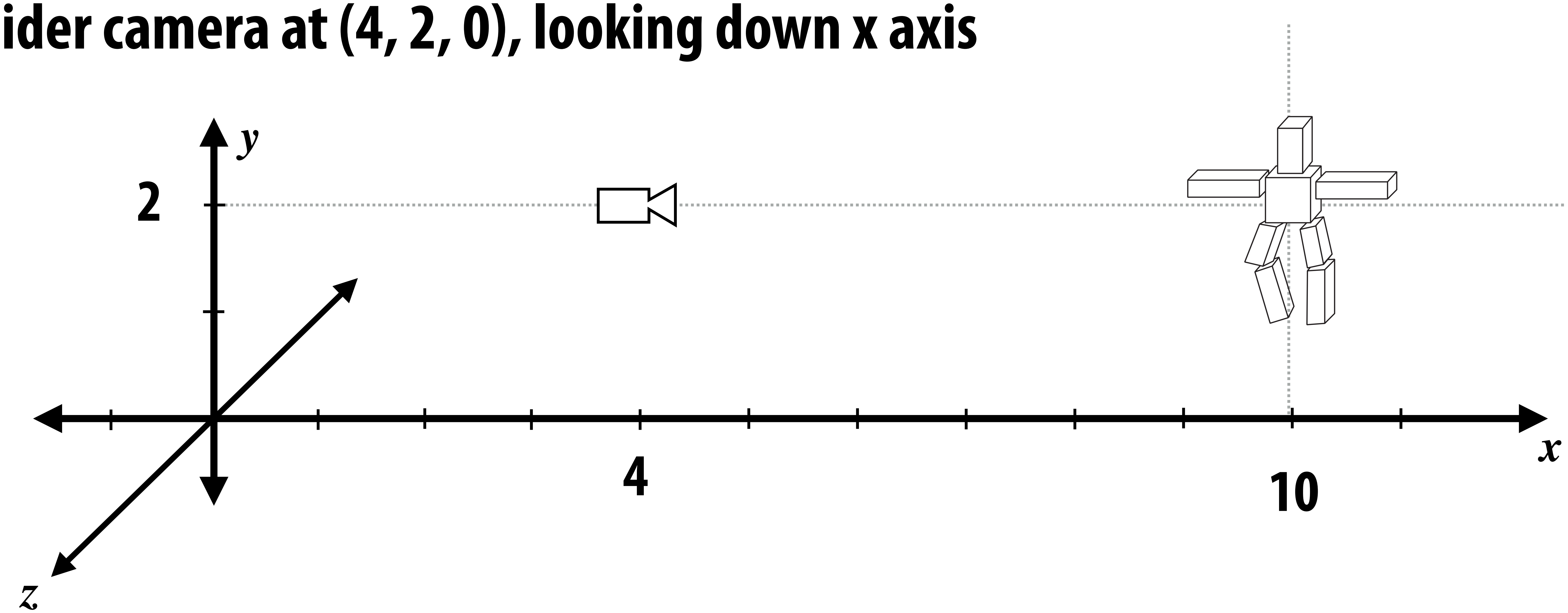
Step 3: scale by (W/2,H/2)

* This slide adopts convention that top-left of screen is (0,0) to match SVG convention in Assignment 1.
Many 3D graphics systems like OpenGL place (0,0) in bottom-left. In this case what would the transform be?

Example: simple camera transform

Consider object positioned in world at (10, 2, 0)

Consider camera at (4, 2, 0), looking down x axis



What transform places in the object in a coordinate space where the camera is at the origin and the camera is looking directly down the -z axis?

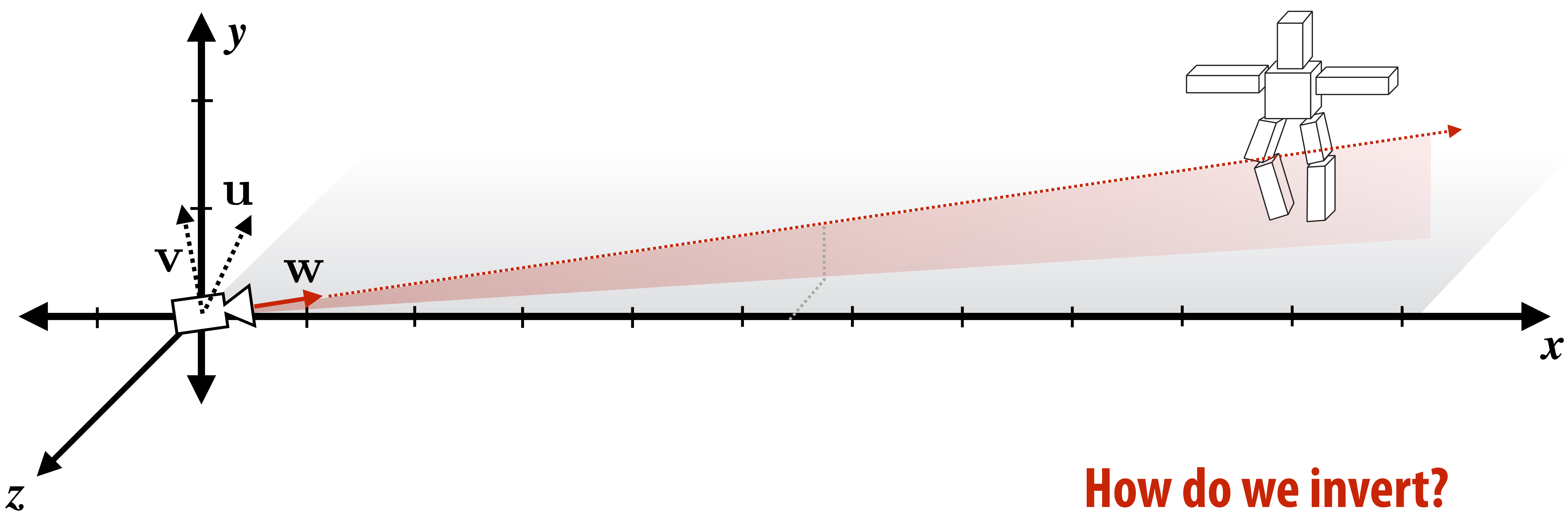
- **Translating object vertex positions by (-4, -2, 0) yields position relative to camera**
- **Rotation about y by $\pi/2$ gives position of object in new coordinate system where camera's view direction is aligned with the -z axis ***

* The convenience of such a coordinate system will become clear on the next slide!

Camera looking in a different direction

Consider camera at origin looking in direction $\mathbf{w}$

What transform places in the object in a coordinate space where the camera is at the origin and the camera is looking directly down the -z axis?



How do we invert?

$$\mathbf{R}^{-1} = \mathbf{R}^T = \begin{bmatrix} \mathbf{u}_x & \mathbf{u}_y & \mathbf{u}_z \\ \mathbf{v}_x & \mathbf{v}_y & \mathbf{v}_z \\ -\mathbf{w}_x & -\mathbf{w}_y & -\mathbf{w}_z \end{bmatrix}$$

Form orthonormal basis around $\mathbf{w}$: (see $\mathbf{u}$ and $\mathbf{v}$)

Consider rotation matrix: $\mathbf{R}$

$$\mathbf{R} = \begin{bmatrix} \mathbf{u}_x & \mathbf{v}_x & -\mathbf{w}_x \\ \mathbf{u}_y & \mathbf{v}_y & -\mathbf{w}_y \\ \mathbf{u}_z & \mathbf{v}_z & -\mathbf{w}_z \end{bmatrix}$$

Why is that the inverse?

$$\mathbf{R}^T\mathbf{u} = \begin{bmatrix}\mathbf{u}\cdot\mathbf{u} & \mathbf{v}\cdot\mathbf{u} & -\mathbf{w}\cdot\mathbf{u}\end{bmatrix}^T = \begin{bmatrix}1 & 0 & 0\end{bmatrix}^T$$

$$\mathbf{R}^T\mathbf{v} = \begin{bmatrix}\mathbf{u}\cdot\mathbf{v} & \mathbf{v}\cdot\mathbf{v} & -\mathbf{w}\cdot\mathbf{v}\end{bmatrix}^T = \begin{bmatrix}0 & 1 & 0\end{bmatrix}^T$$

$$\mathbf{R}^T\mathbf{w} = \begin{bmatrix}\mathbf{u}\cdot\mathbf{w} & \mathbf{v}\cdot\mathbf{w} & -\mathbf{w}\cdot\mathbf{w}\end{bmatrix}^T = \begin{bmatrix}0 & 0 & -1\end{bmatrix}^T$$

$\mathbf{R}$ maps x-axis to $\mathbf{u}$, y-axis to $\mathbf{v}$, z axis to -$\mathbf{w}$

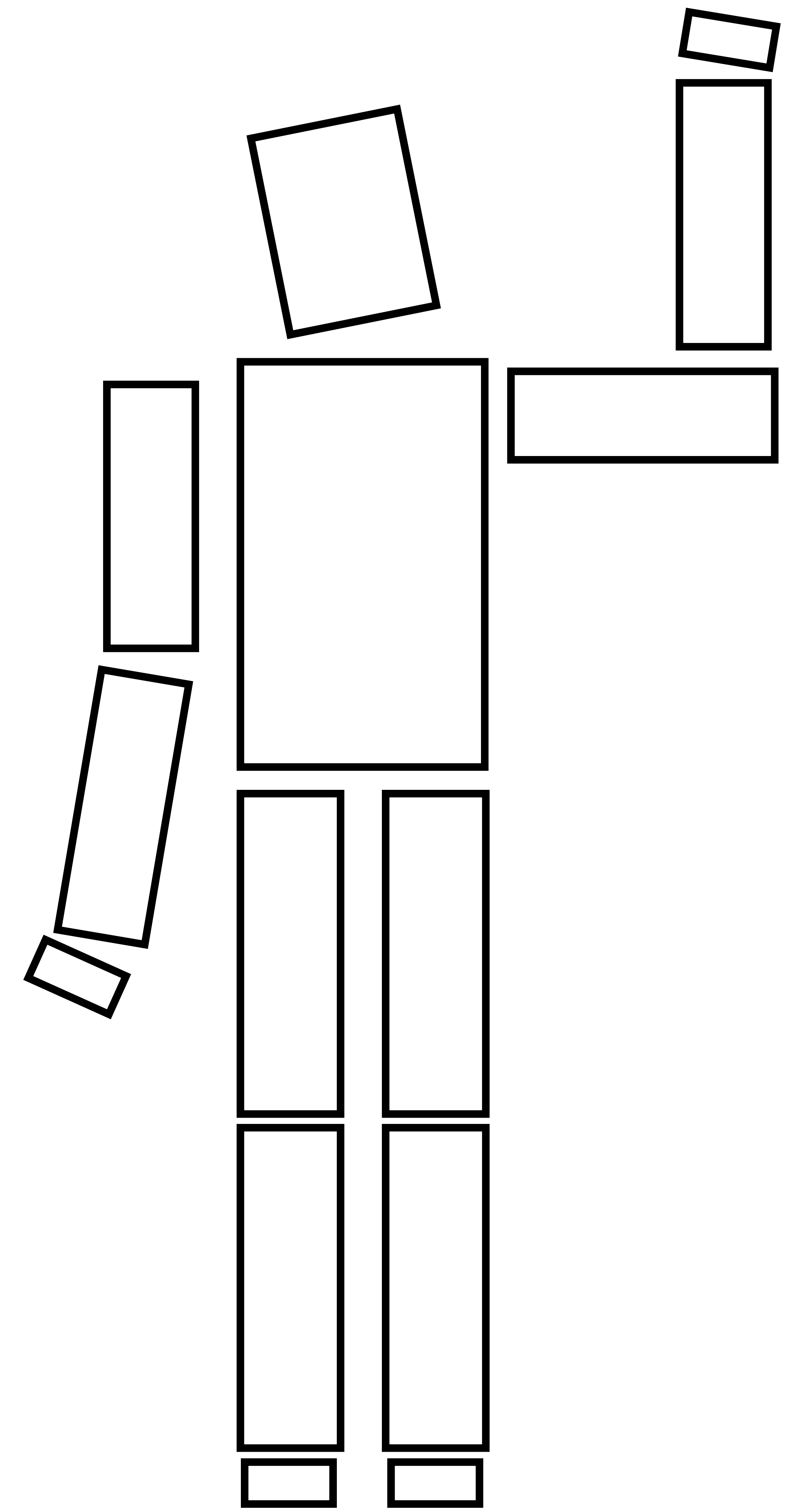
Self-check exercise (for home)

- Given a camera position P
- And a camera orientation given by orthonormal basis u,v,w (camera looking in w)
- What is a transformation matrix that places the scene in a coordinate space where…
  - The camera is at the origin
  - The camera is looking down -z.

Let’s make a cube person

Skeleton - hierarchical representation

torso
- head
- right arm
  - upper arm
    - lower arm
      - hand
- left arm
  - upper arm
    - lower arm
      - hand
- right leg
  - upper leg
    - lower leg
      - foot
- left leg
  - upper leg
    - lower leg
      - foot



Hierarchical representation

- Grouped representation (tree)
  - Each group contains subgroups and/or shapes
  - Each group is associated with a transform relative to parent group
  - Transform on leaf-node shape is concatenation of all transforms on path from root node to leaf
  - Changing a group's transform affects all parts
    - Allows high level editing by changing only one node
    - E.g. raising left arm requires changing only one transform for that group

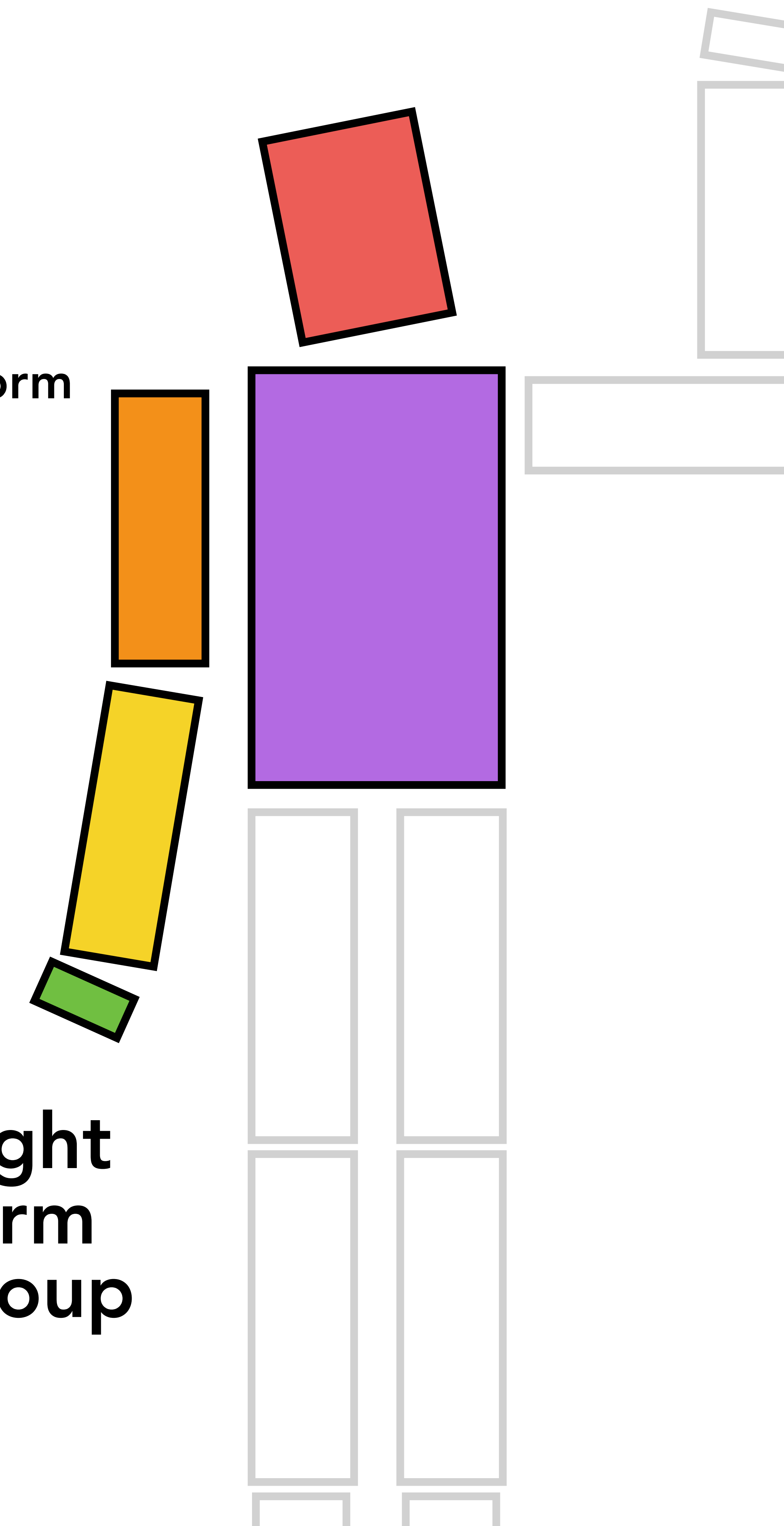
Skeleton - hierarchical representation

```
translate(0, 10);
    drawTorso();
    pushmatrix();  // push a copy of transform onto stack
        translate(0, 5);  // right-multiply onto current transform
        rotate(headRotation);  // right-multiply onto current transform
        drawHead();
    popmatrix();  // pop current transform off stack
    pushmatrix();
        translate(-2, 3);
        rotate(rightShoulderRotation);
        drawUpperArm();
        pushmatrix();
            translate(0, -3);
            rotate(elbowRotation);
            drawLowerArm();
            pushmatrix();
                translate(0, -3);
                rotate(wristRotation);
                drawHand();
            popmatrix();
        popmatrix();
    popmatrix();
    ....
```

right hand

right lower arm group

right arm group



Skeleton - hierarchical representation

```
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        popmatrix();
    popmatrix();
    ....
```
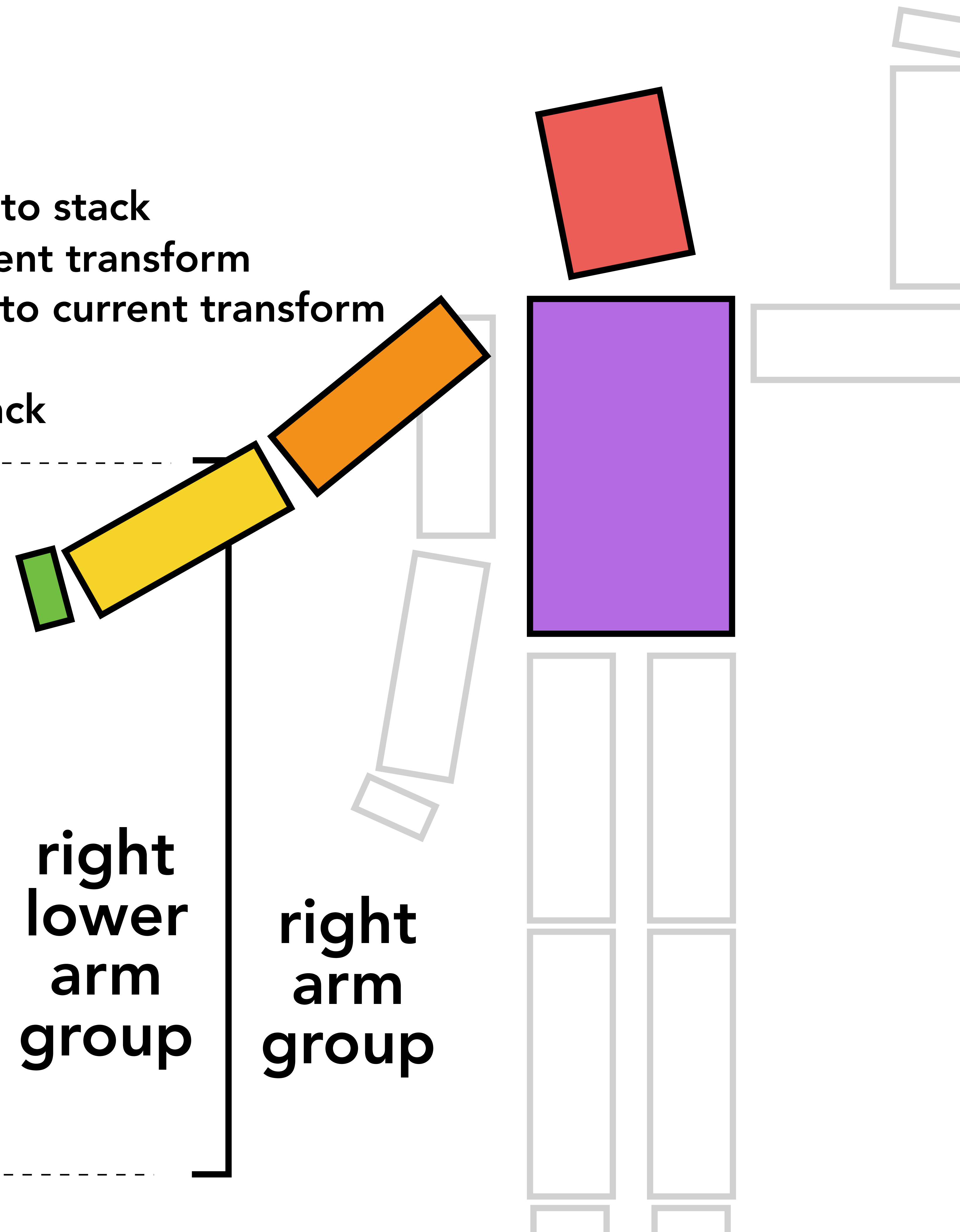
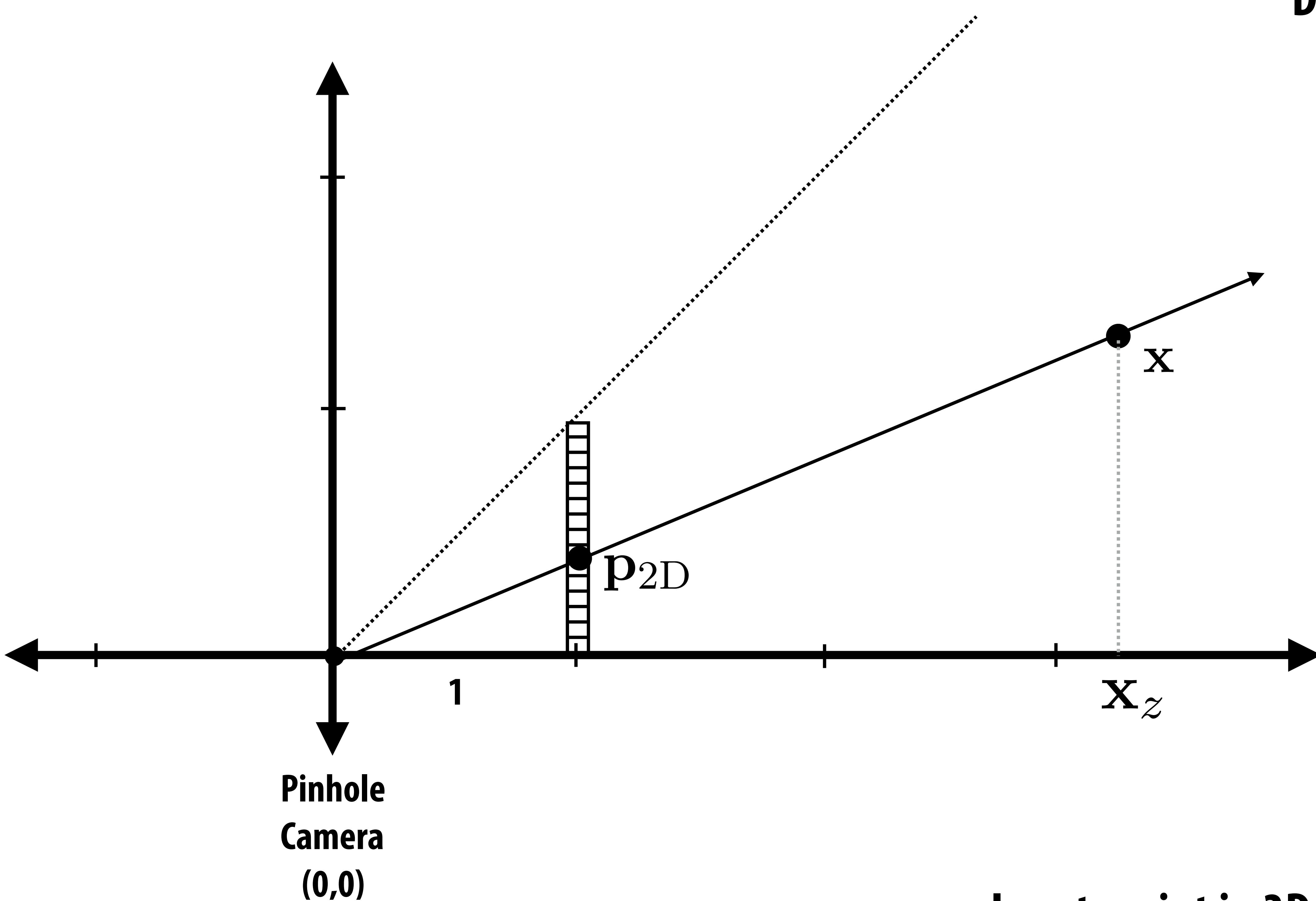

right hand

right lower arm group

right arm group



Basic perspective projection



Desired perspective projected result (2D point):

$$\mathbf{p}_{\mathrm{2D}} = \begin{bmatrix} \mathbf{x}_x/\mathbf{x}_z & \mathbf{x}_y/\mathbf{x}_z \end{bmatrix}^T$$

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Input: point in 3D-H $\mathbf{x} = \begin{bmatrix} \mathbf{x}_x & \mathbf{x}_y & \mathbf{x}_z & 1 \end{bmatrix}$

After applying $\mathbf{P}$: point in 3D-H $\mathbf{P}\mathbf{x} = \begin{bmatrix} \mathbf{x}_x & \mathbf{x}_y & \mathbf{x}_z & \mathbf{x}_z \end{bmatrix}^T$

After homogeneous divide: $\begin{bmatrix} \mathbf{x}_x/\mathbf{x}_z & \mathbf{x}_y/\mathbf{x}_z & 1 \end{bmatrix}^T$

(throw out third component)

Assumption:
Pinhole camera at (0,0) looking down z

More about perspective next lecture!

Transformations summary

- Transformations can be interpreted as operations that move points in space
  - e.g., for modeling, animation
- Or as a change of coordinate system
  - e.g., screen and view transforms
- Construct complex transformations as compositions of basic transforms
- Homogeneous coordinate representation allows for expression of non-linear transforms (e.g., translation, perspective projection) as matrix operations (linear transforms) in higher-dimensional space
  - Matrix representation affords simple implementation and efficient composition